

# HYPERSYMPLECTIC FOUR-DIMENSIONAL LIE ALGEBRAS

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**Abstract.** A study is made of real Lie algebras admitting a hypersymplectic structure, and we provide a method to construct such hypersymplectic Lie algebras. We use this method in order to obtain the classification of all hypersymplectic structures on four-dimensional Lie algebras, and we describe the associated metrics on the corresponding Lie groups.

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## 1. INTRODUCTION

A hypersymplectic structure on a manifold is a complex product structure, i.e. a pair  $\{J, E\}$  of a complex structure and a product structure that anticommute, together with a compatible metric such that the associated 2-forms are closed. This notion is similar to that of a hyperkähler structure, where the base manifold carries a hypercomplex structure, i.e. a pair  $\{J_1, J_2\}$  of anticommuting complex structures.

Hypersymplectic structures were introduced by N. Hitchin in [7], and they are also referred to as *neutral hyperkähler* structures in [8] and as *parahyperkähler* structures in [16]. Hypersymplectic structures on manifolds have become an important subject of study lately, due mainly to its applications in theoretical physics (specially in dimension 4). See for instance [3], where there is a discussion on the relationship between hypersymplectic metrics and the  $N = 2$  superstring. Hypersymplectic metrics on a manifold are Ricci-flat and the associated holonomy group is contained in the real symplectic group.

In [8], H. Kamada determines the compact complex surfaces which admit hypersymplectic structures. These complex surfaces are either complex tori or primary Kodaira surfaces; Kamada also shows when the hypersymplectic metrics on these surfaces are flat. In [5], examples of (non flat) hypersymplectic structures are given on Kodaira manifolds, which are special compact quotients of 2-step nilpotent Lie groups. These hypersymplectic structures are not invariant by the nilpotent Lie group.

The main goal of this paper is to give the classification, up to equivalence, of all left-invariant hypersymplectic structures on 4-dimensional Lie groups. These Lie groups will provide examples of hypersymplectic structures in non compact manifolds, since their underlying differentiable manifolds are diffeomorphic to  $\mathbb{R}^4$ . In order to perform this classification, we begin in §3 the study of hypersymplectic structures on real Lie algebras. We obtain that, associated to a hypersymplectic structure  $\{J, E, g\}$  on a Lie algebra  $\mathfrak{g}$ , there are two 3-tuples  $(\mathfrak{g}_+, \nabla^+, \omega_+)$ ,  $(\mathfrak{g}_-, \nabla^-, \omega_-)$ , where  $\mathfrak{g}_\pm$  are Lie subalgebras of  $\mathfrak{g}$  such that  $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$  and  $\mathfrak{g}_- = J\mathfrak{g}_+$ ,  $\nabla^\pm$  is a flat torsion-free connection on  $\mathfrak{g}_\pm$  and  $\omega_\pm$  is a symplectic form on  $\mathfrak{g}_\pm$  such that  $\omega_+(x, y) = \omega_-(Jx, Jy)$  for all  $x, y \in \mathfrak{g}_+$  and  $\nabla^\pm \omega_\pm = 0$ . Conversely, we show that, in certain cases, given two 3-tuples  $(\mathfrak{u}, \nabla, \omega)$  and  $(\mathfrak{v}, \nabla', \omega')$  satisfying the same conditions as above, we can obtain a hypersymplectic structure on  $\mathfrak{u} \oplus \mathfrak{v}$  (direct sum of vector spaces). This result will be used in the 4-dimensional case. We also deal with equivalences between hypersymplectic structures.

Next, in §4, we give the first steps in order to achieve the classification mentioned above, namely, we determine the flat torsion-free connections on the 2-dimensional Lie algebras which are compatible with a symplectic form and obtain their equivalence classes.

In §5, we prove our main result which states that, aside from the abelian Lie algebra, there are only three Lie algebras which admit a hypersymplectic structure. One of them is a central extension of the 3-dimensional Heisenberg algebra  $\mathfrak{h}_3$ ; the second one is an extension of  $\mathbb{R}^3$  and the third one is an extension of  $\mathfrak{h}_3$ . We also parameterize the underlying complex product structures. In §6, we show that a complex product structure on a Lie algebra admits at most one compatible metric (up to a multiplicative constant), and we determine the hypersymplectic metrics for each of the complex product structures in the Lie algebras obtained previously. We point out the cases when these metrics are flat and/or complete. As an illustration we exhibit the following examples of hypersymplectic metrics on  $\mathbb{R}^4$  with canonical global coordinates  $t, x, y, z$ :

- (i)  $g = dt^2 + dx^2 - dy^2 - dz^2$  (flat and complete).
- (ii)  $g = e^{-t} dt(dz - \frac{1}{2}x dy + \frac{1}{2}y dx) + e^{-t} dx dy$  (flat but not complete).
- (iii)  $g = e^t dt dz + e^{2t} dz^2 - dx dy + e^{2t} dy^2$  (neither flat nor complete).

The three metrics above are hence not isometric.

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## 2. PRELIMINARIES

We start recalling some definitions which will be used throughout this work. All Lie algebras will be finite dimensional and defined over  $\mathbb{R}$ , unless explicitly stated.

For an arbitrary connection  $\nabla$  on (the tangent bundle of) a manifold  $M$ , the torsion and curvature tensor fields  $T$  and  $R$  are defined by

$$\begin{aligned} T(X, Y) &= \nabla_X Y - \nabla_Y X - [X, Y] \\ R(X, Y) &= \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]} \end{aligned}$$

for  $X, Y$  smooth vector fields on  $M$ . The connection is called *torsion-free* when  $T = 0$ , and *flat* when  $R = 0$ .

Let  $G$  be a Lie group with Lie algebra  $\mathfrak{g}$  and suppose that  $G$  admits a left-invariant connection  $\nabla$ . This means that if  $X, Y \in \mathfrak{g}$  are two left-invariant vector fields on  $G$  then  $\nabla_X Y \in \mathfrak{g}$  is also left-invariant. Accordingly, one may define a connection on a Lie algebra  $\mathfrak{g}$  to be merely a  $\mathfrak{g}$ -valued bilinear form  $\mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ . One can speak of the torsion and curvature of such a connection using the formulae above, with brackets determined by the structure of  $\mathfrak{g}$ . It is known that the completeness of the left-invariant connection  $\nabla$  on  $G$  can be studied by considering simply the corresponding connection on the Lie algebra  $\mathfrak{g}$ . Indeed, the connection  $\nabla$  on  $G$  will be (geodesically) complete if and only if the differential equation on  $\mathfrak{g}$

$$(1) \quad \dot{x}(t) = -\nabla_{x(t)} x(t)$$

admits solutions  $x(t) \in \mathfrak{g}$  defined for all  $t \in \mathbb{R}$  (see for instance [6], where only metric connections are considered).

We also recall the definition of complex structures and product structures on a Lie algebra, which are also modelled on the corresponding notions for smooth manifolds.

An *almost complex structure* on a Lie algebra  $\mathfrak{g}$  is a linear endomorphism  $J : \mathfrak{g} \longrightarrow \mathfrak{g}$  satisfying  $J^2 = -\mathbf{1}$ . If  $J$  satisfies the condition

$$(2) \quad J[X, Y] = [JX, Y] + [X, JY] + J[JX, JY] \quad \text{for all } X, Y \in \mathfrak{g},$$

we will say that  $J$  is *integrable* and we will call it a *complex structure* on  $\mathfrak{g}$ . Note that the dimension of a Lie algebra carrying an almost complex structure must be even. We recall that a *hypercomplex structure* on the Lie algebra  $\mathfrak{g}$  is a pair  $\{J_1, J_2\}$  of complex structures on  $\mathfrak{g}$  such that  $J_1 J_2 = -J_2 J_1$ . The dimension of a hypercomplex Lie algebra is a multiple of 4.

Next, an *almost product structure* on  $\mathfrak{g}$  is a linear endomorphism  $E : \mathfrak{g} \longrightarrow \mathfrak{g}$  satisfying  $E^2 = \mathbf{1}$  (and not equal to  $\pm \mathbf{1}$ ). It is said to be *integrable* if

$$(3) \quad E[X, Y] = [EX, Y] + [X, EY] - E[EX, EY] \quad \text{for all } X, Y \in \mathfrak{g}.$$

An integrable almost product structure will be called a *product structure*. If  $\mathfrak{g}_\pm$  is the eigenspace of  $E$  associated to the eigenvalue  $\pm 1$  of  $E$ , then the integrability of  $E$  is equivalent to the fact of  $\mathfrak{g}_\pm$  being Lie subalgebras of  $\mathfrak{g}$ . If  $\dim \mathfrak{g}_+ = \dim \mathfrak{g}_-$ , the product structure  $E$  is called a *paracomplex structure* [9, 10]. In this case,  $\mathfrak{g}$  also has even dimension.

An appropriate combination of these two structures on Lie algebras is called a complex product structure, and its definition is given below. This new structure is similar to a hypercomplex structure, where one of the complex structures has been replaced by a product structure.

**Definition 2.1.** A *complex product structure* on the Lie algebra  $\mathfrak{g}$  is a pair  $\{J, E\}$  of a complex structure  $J$  and a product structure  $E$  satisfying  $JE = -EJ$ .

Complex product structures on Lie algebras have been studied in [2], from where we recall some of their main properties. The condition  $JE = -EJ$  implies that  $J$  is an isomorphism (as vector spaces) between the eigenspaces  $\mathfrak{g}_+$  and  $\mathfrak{g}_-$  corresponding to the eigenvalues  $+1$  and  $-1$  of  $E$ , respectively; thus,  $E$  is in fact a paracomplex structure on  $\mathfrak{g}$ . Every complex product structure on  $\mathfrak{g}$  has therefore an associated *double Lie algebra*  $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ , i.e.,  $\mathfrak{g}_+$  and  $\mathfrak{g}_-$  are Lie subalgebras of  $\mathfrak{g}$  such that  $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$  (direct sum of vector spaces) and  $\mathfrak{g}_- = J\mathfrak{g}_+$ , where  $E|_{\mathfrak{g}_+} = \mathbf{1}$ ,  $E|_{\mathfrak{g}_-} = -\mathbf{1}$ . We note that the dimension of a Lie algebra with a complex product structure is even but it need not be a multiple of 4.

The complex product structure  $\{J, E\}$  on  $\mathfrak{g}$  determines uniquely a torsion-free connection  $\nabla^{\text{CP}}$  on  $\mathfrak{g}$  such that  $\nabla^{\text{CP}} J = \nabla^{\text{CP}} E = 0$ , where this equations mean that

$$\nabla_x^{\text{CP}} Jy = J\nabla_x^{\text{CP}} y, \quad \nabla_x^{\text{CP}} Ey = E\nabla_x^{\text{CP}} y$$

for all  $x, y \in \mathfrak{g}$ . As a consequence, we note that  $\nabla_x^{\text{CP}} y \in \mathfrak{g}_\pm$  for any  $x \in \mathfrak{g}$  and  $y \in \mathfrak{g}_\pm$ . Take now  $x \in \mathfrak{g}_+, y \in \mathfrak{g}_-$ . Since  $\nabla^{\text{CP}}$  has no torsion, we obtain that

$$(4) \quad [x, y] = -\nabla_y^{\text{CP}} x + \nabla_x^{\text{CP}} y \in \mathfrak{g}_+ \oplus \mathfrak{g}_-$$

is the decomposition of  $[x, y]$  into components, according to the splitting  $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ . The connection  $\nabla^{\text{CP}}$  restricts to *flat* torsion-free connections on  $\mathfrak{g}_+$  and  $\mathfrak{g}_-$ , say  $\nabla^+$  and  $\nabla^-$ , respectively. Thus, we have that  $\nabla^{\text{CP}}$  is flat if and only if  $R(x_+, x_-) = 0$  for all  $x_+ \in \mathfrak{g}_+, x_- \in \mathfrak{g}_-$ , where  $R$  is the curvature of  $\nabla^{\text{CP}}$ . We recall that flat torsion-free connections on a Lie algebra are also known as “left-symmetric algebra” (LSA) structures.

### 3. HYPERSYMPLECTIC STRUCTURES ON LIE ALGEBRAS

We have observed the close resemblance between complex product structures and hypercomplex structures on Lie algebras. Bearing this similarity in mind, we study in this section a special kind of metrics on a Lie algebra with a complex product structure, just as hyperkähler metrics appear in the context of hypercomplex structures.

Let  $\mathfrak{g}$  be a Lie algebra endowed with a complex product structure  $\{J, E\}$  and let  $g$  be a metric on  $\mathfrak{g}$ , i.e.,  $g$  is a non degenerate symmetric bilinear form  $g : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{R}$ . We will say that  $g$  is *compatible* with the complex product structure if

$$(5) \quad g(Jx, Jy) = g(x, y), \quad g(Ex, Ey) = -g(x, y)$$

for all  $x, y \in \mathfrak{g}$ .

Let  $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$  denote the double Lie algebra associated to the complex product structure  $\{J, E\}$ , where  $\mathfrak{g}_- = J\mathfrak{g}_+$  and let  $g$  be a compatible metric. Then the subalgebras  $\mathfrak{g}_+$  and  $\mathfrak{g}_-$  are isotropic subspaces of  $\mathfrak{g}$  with respect to  $g$ , i.e.,

$$g(\mathfrak{g}_+, \mathfrak{g}_+) = 0, \quad g(\mathfrak{g}_-, \mathfrak{g}_-) = 0.$$

For if  $x, y \in \mathfrak{g}_+$ , we have that  $g(Ex, Ey) = g(x, y)$  because of the definition of  $\mathfrak{g}_+$ . But  $g(Ex, Ey) = -g(x, y)$  due to (5). Hence  $g(x, y) = 0$  for  $x, y \in \mathfrak{g}_+$  and equally for  $x, y \in \mathfrak{g}_-$ . From this it is clear that  $\mathfrak{g}_+^\perp = \mathfrak{g}_+$  and  $\mathfrak{g}_-^\perp = \mathfrak{g}_-$  and also that the signature of  $g$  is  $(m, m)$ , where  $\dim \mathfrak{g} = 2m$ .

Let us now define the following bilinear forms on  $\mathfrak{g}$ :

$$(6) \quad \omega_1(x, y) = g(Jx, y), \quad \omega_2(x, y) = g(Ex, y), \quad \omega_3(x, y) = g(JEx, y)$$

for  $x, y \in \mathfrak{g}$ . Using (5), it is readily verified that these forms are in fact skew-symmetric, so that  $\omega_i \in \bigwedge^2 \mathfrak{g}^*$  for  $i = 1, 2, 3$ . Note that these forms are non degenerate, since  $g$  is non degenerate and  $J$  and  $E$  are isomorphisms. In the following result we show the existing relationships between these 2-forms on  $\mathfrak{g}$  and the decomposition of this Lie algebra induced by the product structure  $E$ .

**Lemma 3.1.** *The 2-forms  $\omega_i$ ,  $i = 1, 2, 3$ , on  $\mathfrak{g}$  satisfy the following properties:*

- (i)  $\omega_1(x, y) = \omega_1(Jx, Jy) = \omega_1(Ex, Ey)$  for all  $x, y \in \mathfrak{g}$ , whence  $\omega_1(x, y) = 0$  for  $x \in \mathfrak{g}_+, y \in \mathfrak{g}_-$ .
- (ii)  $\omega_2(x, y) = -\omega_2(Jx, Jy) = -\omega_2(Ex, Ey)$  for any  $x, y \in \mathfrak{g}$ , whence  $\omega_2(x, y) = 0$  for  $x, y \in \mathfrak{g}_+$  or  $x, y \in \mathfrak{g}_-$ .
- (iii)  $\omega_3(x, y) = -\omega_3(Jx, Jy) = \omega_3(Ex, Ey)$  for all  $x, y \in \mathfrak{g}$ , whence  $\omega_3(x, y) = 0$  for  $x \in \mathfrak{g}_+, y \in \mathfrak{g}_-$ .

*Proof.* The proof is straightforward. □

Let  $\omega_+$  and  $\omega_-$  denote the restriction of  $\omega_1$  to  $\mathfrak{g}_+$  and  $\mathfrak{g}_-$ , respectively. From (i) of the previous lemma and the fact that  $\omega_1$  is non degenerate, it is easy to see that both  $\omega_+$  and  $\omega_-$  are non degenerate. Hence,  $m = \dim \mathfrak{g}_+ = \dim \mathfrak{g}_-$  must be an even number, say  $m = 2n$ , and therefore  $\dim \mathfrak{g} = 4n$  and the signature of  $g$  is  $(2n, 2n)$ .

From Lemma 3.1 (i), we obtain that

$$(7) \quad \omega_+(x, y) = \omega_-(Jx, Jy)$$

for all  $x, y \in \mathfrak{g}_+$ . We shall show next that the forms  $\omega_1$ ,  $\omega_2$  and  $\omega_3$  can be written in terms exclusively of  $\omega_+$ . In fact, we will show that

$$(8) \quad \omega_1(x + Jx', y + Jy') = \omega_+(x, y) + \omega_+(x', y'),$$

$$(9) \quad \omega_2(x + Jx', y + Jy') = -\omega_+(x, y') + \omega_+(y, x'),$$

$$(10) \quad \omega_3(x + Jx', y + Jy') = \omega_+(x, y) - \omega_+(x', y')$$

for all  $x, y, x', y' \in \mathfrak{g}_+$ . Indeed, to prove (8) we compute

$$\begin{aligned} \omega_1(x + Jx', y + Jy') &= \omega_1(x, y) + \omega_1(Jx', Jy') \\ &= \omega_1(x, y) + \omega_1(x', y') \\ &= \omega_+(x, y) + \omega_+(x', y') \end{aligned}$$

using Lemma 3.1 (i). Next, we note that, for any  $u, v \in \mathfrak{g}_+$  we have

$$\omega_2(u, Jv) = g(Eu, Jv) = -g(JEu, v) = -g(Ju, v) = -\omega_1(u, v).$$

Now, in order to prove (9) we compute

$$\begin{aligned} \omega_2(x + Jx', y + Jy') &= \omega_2(x, Jy') + \omega_2(Jx', y) \\ &= -\omega_1(x, y') + \omega_1(y, x') \\ &= -\omega_+(x, y') + \omega_+(y, x') \end{aligned}$$

because of Lemma 3.1 (ii). Finally, to verify that (10) holds, we note first that from (6) we see that  $\omega_3(u, v) = \omega_1(u, v)$  for  $u, v \in \mathfrak{g}_+$ ; as a consequence, we have

$$\begin{aligned} \omega_3(x + Jx', y + Jy') &= \omega_3(x, y) + \omega_3(Jx', Jy') \\ &= \omega_3(x, y) - \omega_3(x', y') \\ &= \omega_1(x, y) - \omega_1(x', y') \\ &= \omega_+(x, y) - \omega_+(x', y'), \end{aligned}$$

using Lemma 3.1 (iii).

Let us recall now that given a 2-form  $\omega$  on a Lie algebra  $\mathfrak{g}$ , there is an associated 3-form  $d\omega \in \bigwedge^3 \mathfrak{g}^*$  given by

$$(d\omega)(x, y, z) = -\omega([x, y], z) + \omega([x, z], y) - \omega([y, z], x)$$

for all  $x, y, z \in \mathfrak{g}$ . The 2-form  $\omega$  is called *closed* if  $d\omega = 0$ ; if  $\omega$  is non degenerate and closed, it is called a *symplectic form* on  $\mathfrak{g}$ .

Naturally, we are mainly interested in the case when all of the 2-forms given in (6) are closed and hence symplectic. We introduce therefore the following definition, equivalent to the one given by N. Hitchin in [7].

**Definition 3.2.** Let  $\{J, E\}$  be a complex product structure on the Lie algebra  $\mathfrak{g}$  and let  $g$  be a metric on  $\mathfrak{g}$  compatible with  $\{J, E\}$ . If the 2-forms  $\omega_i \in \bigwedge^2 \mathfrak{g}^*$  defined in (6) are closed, we will say that  $\{J, E, g\}$  is a *hypersymplectic structure* on  $\mathfrak{g}$ . The Lie algebra  $\mathfrak{g}$  will be referred to as a *hypersymplectic Lie algebra* and  $g$  will be called a *hypersymplectic metric*.

The surprising fact is that if one of the 2-forms  $\omega_1$  or  $\omega_3$  is closed, then all three of these 2-forms are closed, as the following result shows.

**Proposition 3.3.** *Let  $\{J, E\}$  be a complex product structure on  $\mathfrak{g}$  with associated double Lie algebra  $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ . Let  $\nabla^+$  and  $\nabla^-$  denote the flat torsion-free connections on  $\mathfrak{g}_+$  and  $\mathfrak{g}_-$  induced by  $\nabla^{\text{CP}}$ . Suppose  $g$  is a compatible metric on  $\mathfrak{g}$  and let  $\omega_i$ ,  $i = 1, 2, 3$ , be the 2-forms on  $\mathfrak{g}$  given by (6) and  $\omega_+$  and  $\omega_-$  be as above. Then the following statements are equivalent:*

- (i)  $\omega_1$  is closed;
- (ii)  $\omega_3$  is closed;
- (iii)  $\nabla^+\omega_+ = 0$  and  $\nabla^-\omega_- = 0$ .

Furthermore, if one of the conditions above holds, then

- (iv)  $\omega_2$  is closed.

*Remark.* We recall that a 2-form  $\omega$  on a Lie algebra  $\mathfrak{h}$  is parallel with respect to a connection  $\nabla$  on  $\mathfrak{h}$ , i.e.  $\nabla\omega = 0$ , if the condition

$$\omega(\nabla_x y, z) = \omega(\nabla_x z, y)$$

holds for all  $x, y, z \in \mathfrak{h}$ .

*Proof.* (i)  $\Leftrightarrow$  (iii) Let us suppose first that (i) holds. For  $x, y \in \mathfrak{g}_+$  and  $z = Ju$  with  $u \in \mathfrak{g}_+$  we have that

$$\begin{aligned} (11) \quad (d\omega_1)(x, y, Ju) &= \omega_1([x, Ju], y) - \omega_1([y, Ju], x) \\ &= -\omega_1(\nabla_{Ju}^{\text{CP}} x, y) + \omega_1(\nabla_{Ju}^{\text{CP}} y, x) \\ &= -\omega_1(\nabla_{Ju}^{\text{CP}} Jx, Jy) + \omega_1(\nabla_{Ju}^{\text{CP}} Jy, Jx) \\ &= -\omega_-(\nabla_{Ju}^- Jx, Jy) + \omega_-(\nabla_{Ju}^- Jy, Jx), \end{aligned}$$

using (4). As  $d\omega_1 = 0$ , we obtain that  $\nabla^-\omega_- = 0$ . If we consider now  $x \in \mathfrak{g}_+$  and  $y = Jv$ ,  $z = Ju$  with  $u, v \in \mathfrak{g}_+$ , we have that

$$\begin{aligned} (12) \quad (d\omega_1)(x, Jv, Ju) &= -\omega_1([x, Jv], Ju) + \omega_1([x, Ju], Jv) \\ &= -\omega_1(\nabla_x^{\text{CP}} Jv, Ju) + \omega_1(\nabla_x^{\text{CP}} Ju, Jv) \\ &= -\omega_1(\nabla_x^{\text{CP}} v, u) + \omega_1(\nabla_x^{\text{CP}} u, v) \\ &= -\omega_+(\nabla_x^+ v, u) + \omega_+(\nabla_x^+ u, v), \end{aligned}$$

using again (4). As  $d\omega_1 = 0$ , we obtain that  $\nabla^+\omega_+ = 0$ . Thus, (iii) holds.

Conversely, let us suppose that (iii) holds. We note first that, as  $\nabla^+$  and  $\nabla^-$  are torsion-free, one obtains that  $d\omega_+ = 0$  and  $d\omega_- = 0$ . Suppose that  $x, y, z \in \mathfrak{g}_+$ . Then  $d\omega_1(x, y, z) = d\omega_+(x, y, z) = 0$  since  $\omega_+$  is closed. Similarly, for  $x, y, z \in \mathfrak{g}_-$ , we have  $d\omega_1(x, y, z) = d\omega_-(x, y, z) = 0$  since  $\omega_-$  is closed. Now, if  $x, y \in \mathfrak{g}_+$  and  $z = Ju$  with  $u \in \mathfrak{g}_+$ , from equations (11) and  $\nabla^-\omega_- = 0$ , we have that  $(d\omega_1)(x, y, Ju) = 0$ . Next, if  $x \in \mathfrak{g}_+$  and  $y = Jv$ ,  $z = Ju$  with  $u, v \in \mathfrak{g}_+$ , from equations (12) and  $\nabla^+\omega_+ = 0$ , we have that  $(d\omega_1)(x, Jv, Ju) = 0$ . Therefore,  $d\omega_1 = 0$ .

- (ii)  $\Leftrightarrow$  (iii) The proof is similar to the proof of (i)  $\Leftrightarrow$  (iii).

(iii)  $\Rightarrow$  (iv) If  $x, y, z \in \mathfrak{g}_+$  or  $x, y, z \in \mathfrak{g}_-$ , then  $(d\omega_2)(x, y, z) = 0$ , because of Lemma 3.1 (ii). If  $x, y \in \mathfrak{g}_+$  and  $z = Ju$  with  $u \in \mathfrak{g}_+$ , then

$$\begin{aligned} (d\omega_2)(x, y, Ju) &= -\omega_2([x, y], Ju) + \omega_2([x, Ju], y) - \omega_2([y, Ju], x) \\ &= -\omega_2([x, y], Ju) + \omega_2(\nabla_x^{\text{CP}} Ju, y) - \omega_2(\nabla_y^{\text{CP}} Ju, x) \\ &= \omega_1([x, y], u) - \omega_1(\nabla_x^{\text{CP}} u, y) + \omega_1(\nabla_y^{\text{CP}} u, x) \\ &= \omega_+([x, y], u) - \omega_+(\nabla_x^+ u, y) + \omega_+(\nabla_y^+ u, x) \\ &= \omega_+([x, y], u) - \omega_+(\nabla_x^+ y, u) + \omega_+(\nabla_y^+ x, u) \\ &= 0 \end{aligned}$$

because of (4), (iii) and since  $\nabla^+$  is torsion-free. Now suppose that  $x \in \mathfrak{g}_+$  and  $y = Jv$ ,  $z = Ju$  with  $u, v \in \mathfrak{g}_+$ . We have that

$$\begin{aligned} d\omega_2(x, Jv, Ju) &= -\omega_2([x, Jv], Ju) + \omega_2([x, Ju], Jv) - \omega_2([Jv, Ju], x) \\ &= \omega_2(\nabla_{Jv}^{\text{CP}} x, Ju) - \omega_2(\nabla_{Ju}^{\text{CP}} x, Jv) - \omega_2(J[Jv, Ju], Jx) \\ &= -\omega_1(\nabla_{Jv}^{\text{CP}} x, u) + \omega_1(\nabla_{Ju}^{\text{CP}} x, v) - \omega_1(J[Jv, Ju], x) \\ &= -\omega_1(\nabla_{Jv}^{\text{CP}} Jx, Ju) + \omega_1(\nabla_{Ju}^{\text{CP}} Jx, Jv) + \omega_1([Jv, Ju], Jx) \\ &= -\omega_1(\nabla_{Jv}^{\text{CP}} Ju, Jx) + \omega_1(\nabla_{Ju}^{\text{CP}} Jv, Jx) + \omega_1([Jv, Ju], Jx) \\ &= -\omega_-(\nabla_{Jv}^- Ju, Jx) + \omega_-(\nabla_{Ju}^- Jv, Jx) + \omega_1([Jv, Ju], Jx) \\ &= 0 \end{aligned}$$

because of (4), (iii) and since  $\nabla^-$  is torsion-free. Hence, (iv) holds.  $\square$

As a consequence, if, for a metric  $g$  on  $\mathfrak{g}$  compatible with the complex product structure  $\{J, E\}$ , the Kähler form  $\omega_1$  is closed (and hence symplectic), then  $\{J, E, g\}$  is a hypersymplectic structure on  $\mathfrak{g}$ .

Summing up, we have the following result, which describes the structure of a Lie algebra admitting a hypersymplectic structure.

**Theorem 3.4.** *Let  $\{J, E, g\}$  be a hypersymplectic structure on  $\mathfrak{g}$  and let  $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$  be the double Lie algebra associated to the complex product structure  $\{J, E\}$ . Then  $\mathfrak{g}_+$  carries a flat torsion-free connection  $\nabla^+$  and a compatible symplectic form  $\omega_+$ , and similarly,  $\mathfrak{g}_-$  carries a flat torsion-free connection  $\nabla^-$  and a compatible symplectic form  $\omega_-$ . These symplectic forms are related by:  $\omega_+(x, y) = \omega_-(Jx, Jy)$  for  $x, y \in \mathfrak{g}_+$ .*

In the next result we exhibit a converse for Theorem 3.4, showing a method to produce hypersymplectic Lie algebras beginning with two Lie algebras equipped with compatible flat torsion-free connections and symplectic forms. Even though the hypotheses appearing in the statement of the theorem seem rather complicated, we will be able to use this result to obtain all the hypersymplectic 4-dimensional Lie algebras. This will be done in subsequent sections.

**Theorem 3.5.** *Consider the following data:*

- (1)  $\mathfrak{u}$  is a Lie algebra equipped with a flat torsion-free connection  $\nabla$  and a symplectic form  $\omega$  such that  $\nabla\omega = 0$ .

- (2)  $\mathfrak{v}$  is a Lie algebra equipped with a flat torsion-free connection  $\nabla'$  and a symplectic form  $\omega'$  such that  $\nabla'\omega' = 0$ .
- (3) There exists a linear isomorphism  $\varphi : \mathfrak{u} \longrightarrow \mathfrak{v}$  such that
- (i) the representations  $\rho : \mathfrak{u} \longrightarrow \mathfrak{gl}(\mathfrak{v})$  and  $\mu : \mathfrak{v} \longrightarrow \mathfrak{gl}(\mathfrak{u})$  defined by

$$\rho(x)a = \varphi \nabla_x \varphi^{-1}(a), \quad \mu(a)x = \varphi^{-1} \nabla'_a \varphi(x)$$

satisfy

$$\begin{aligned} \rho(x)[a, b] - [\rho(x)a, b] - [a, \rho(x)b] + \rho(\mu(a)x)b - \rho(\mu(b)x)a &= 0, \\ \mu(a)[x, y] - [\mu(a)x, y] - [x, \mu(a)y] + \mu(\rho(x)a)y - \mu(\rho(y)a)x &= 0, \end{aligned}$$

for all  $x, y \in \mathfrak{u}$  and  $a, b \in \mathfrak{v}$ .

- (ii)  $\omega(x, y) = \omega'(\varphi(x), \varphi(y))$  for all  $x, y \in \mathfrak{u}$ .

In this situation the vector space  $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{v}$  admits a Lie bracket extending the Lie brackets on  $\mathfrak{u}$  and  $\mathfrak{v}$  and there is a hypersymplectic structure on  $\mathfrak{g}$  such that  $\mathfrak{g}_+ = \mathfrak{u}$  and  $\mathfrak{g}_- = \mathfrak{v}$ .

*Proof.* Condition (i) in the statement means that  $(\mathfrak{u}, \mathfrak{v}, \rho, \mu)$  is a *matched pair* of Lie algebras (see [11] or [12]). Thus, the bracket on  $\mathfrak{g}$  given by

$$[(x, a), (y, b)] = ([x, y] + \mu(a)y - \mu(b)x, [a, b] + \rho(x)b - \rho(y)a),$$

for  $x, y \in \mathfrak{u}$  and  $a, b \in \mathfrak{v}$  satisfies the Jacobi identity;  $\mathfrak{g}$  with this Lie algebra structure will be denoted  $\mathfrak{g} = \mathfrak{u} \bowtie \mathfrak{v}$  and will be called the *bicrossproduct* of  $\mathfrak{u}$  and  $\mathfrak{v}$ . Observe that  $\mathfrak{u}$  and  $\mathfrak{v}$  are Lie subalgebras of  $\mathfrak{g}$ . If we take into account the definition of  $\rho$  and  $\mu$ , we get that the Lie bracket between an element of  $\mathfrak{u}$  and one of  $\mathfrak{v}$  is given by

$$[(x, 0), (0, a)] = (-\varphi^{-1} \nabla'_a \varphi(x), \varphi \nabla_x \varphi^{-1}(a))$$

for  $x \in \mathfrak{u}$  and  $a \in \mathfrak{v}$ . It has already been proved in [2] that  $\mathfrak{g} = \mathfrak{u} \bowtie \mathfrak{v}$  admits a complex product structure  $\{J, E\}$ , where the endomorphisms  $J$  and  $E$  are defined by

$$J(x, a) = (-\varphi^{-1}(a), \varphi(x)), \quad E|_{\mathfrak{u}} = \mathbf{1}, \quad E|_{\mathfrak{v}} = -\mathbf{1},$$

for  $x \in \mathfrak{u}$ ,  $a \in \mathfrak{v}$ . Furthermore, if  $\nabla^{\text{CP}}$  denotes the torsion-free connection associated to  $\{J, E\}$ , then the restrictions of  $\nabla^{\text{CP}}$  to  $\mathfrak{g}_+ = \mathfrak{u}$  and  $\mathfrak{g}_- = \mathfrak{v}$  are precisely the original connections  $\nabla$  and  $\nabla'$ , respectively.

We proceed now to define a metric  $g$  on  $\mathfrak{g}$  which will be shown to be hypersymplectic. Let  $g$  be given by

$$g(\mathfrak{u}, \mathfrak{u}) = 0, \quad g(\mathfrak{v}, \mathfrak{v}) = 0, \quad g((x, a), (y, b)) = \omega(\varphi^{-1}(b), x) + \omega(\varphi^{-1}(a), y)$$

for  $x, y \in \mathfrak{u}$ ,  $a, b \in \mathfrak{v}$ . It is clear that  $g$  is a metric on  $\mathfrak{g}$ . We should check now that it satisfies (5). We begin with

$$\begin{aligned} g(J(x, a), J(y, b)) &= g((-\varphi^{-1}(a), \varphi(x)), (-\varphi^{-1}(b), \varphi(y))) \\ &= -\omega(y, \varphi^{-1}(a)) - \omega(x, \varphi^{-1}(b)) \\ &= \omega(\varphi^{-1}(a), y) + \omega(\varphi^{-1}(b), x) \\ &= g((x, a), (y, b)) \end{aligned}$$



and now

$$\begin{aligned}
 g(E(x, a), E(y, b)) &= g((x, -a), (y, -b)) \\
 &= \omega(-\varphi^{-1}(b), x) + \omega(-\varphi^{-1}(a), y) \\
 &= -\omega(\varphi^{-1}(b), x) - \omega(\varphi^{-1}(a), y) \\
 &= -g((x, a), (y, b)).
 \end{aligned}$$

Thus (5) holds and  $g$  is compatible with  $\{J, E\}$ .

To see that with this metric we obtain a hypersymplectic structure on  $\mathfrak{g}$ , we only have to see that (iii) of Proposition 3.3 holds. Let us determine firstly the 2-form  $\omega_1$  on  $\mathfrak{g}$ :

$$\begin{aligned}
 \omega_1((x, a), (y, b)) &= g(J(x, a), (y, b)) \\
 &= g((-\varphi^{-1}(a), \varphi(x)), (y, b)) \\
 &= \omega(\varphi^{-1}(b), -\varphi^{-1}(a)) + \omega(x, y) \\
 &= \omega(x, y) + \omega'(a, b).
 \end{aligned}$$

Therefore, the restrictions of  $\omega_1$  to  $\mathfrak{g}_+ = \mathfrak{u}$  and  $\mathfrak{g}_- = \mathfrak{v}$  are precisely the original symplectic forms  $\omega$  and  $\omega'$ , respectively. As  $\nabla^+ \omega_+ = \nabla \omega = 0$  and  $\nabla^- \omega_- = \nabla' \omega' = 0$ , we have that  $g$  is a hypersymplectic metric on  $\mathfrak{g}$ .  $\square$

Any metric  $g$  on a Lie algebra  $\mathfrak{g}$  determines by left-translations a left-invariant metric on  $G$ , where  $G$  is the only simply connected Lie group with  $L(G) = \mathfrak{g}$ . It is easy to verify that the Levi-Civita connection on the manifold  $G$  is also left-invariant, and hence it is determined by its values at  $\mathfrak{g} \cong T_e G$ . Therefore, the metric  $g$  on  $\mathfrak{g}$  determines a connection  $\nabla^g$  on  $\mathfrak{g}$ , also called the Levi-Civita connection associated to  $g$ . This Levi-Civita connection is the only connection on  $\mathfrak{g}$  such that (i) it is torsion-free, and (ii) the endomorphisms  $\nabla_x^g, x \in \mathfrak{g}$ , are skew-adjoint with respect to  $g$ . Just as in the positive definite case, in the neutral setting one can prove the following equivalences:

**Proposition 3.6.** *Let  $\mathfrak{g}$  be a Lie algebra with a complex product structure  $\{J, E\}$  and a compatible metric  $g$ . Let  $\nabla^g$  denote the Levi-Civita connection on  $\mathfrak{g}$  associated to  $g$  and let  $\omega_i, i = 1, 2, 3$ , be the 2-forms on  $\mathfrak{g}$  given in (6). Then the following statements are equivalent:*

- (i) *The metric  $g$  is hypersymplectic, i.e.,  $d\omega_i = 0$  for  $i = 1, 2, 3$ .*
- (ii) *The endomorphisms  $J$  and  $E$  are  $\nabla^g$ -parallel:  $\nabla^g J = \nabla^g E = 0$ .*
- (iii) *The 2-forms  $\omega_i, i = 1, 2, 3$ , are  $\nabla^g$ -parallel:  $\nabla^g \omega_i = 0$  for  $i = 1, 2, 3$ .*

On any Lie algebra  $\mathfrak{g}$  with a hypersymplectic structure  $\{J, E, g\}$  we have canonically defined two torsion-free connections: the connection  $\nabla^{\text{CP}}$  determined by the complex product structure  $\{J, E\}$  and the Levi-Civita connection  $\nabla^g$  corresponding to  $g$ . Recalling that  $\nabla^{\text{CP}}$  is the only torsion-free connection such that  $J$  and  $E$  are parallel, and taking into account the equivalence (i)  $\Leftrightarrow$  (ii) of Proposition 3.6, we obtain that  $\nabla^g = \nabla^{\text{CP}}$ .

We consider now the question of equivalences between hypersymplectic structures. We have the following definition.

**Definition 3.7.** Let  $\{J, E, g\}$  and  $\{J', E', g'\}$  be hypersymplectic structures on the Lie algebras  $\mathfrak{g}$  and  $\mathfrak{g}'$  respectively. These structures are said to be *equivalent* if there exists a

Lie algebra isomorphism  $\xi : \mathfrak{g} \longrightarrow \mathfrak{g}'$  such that

$$(13) \quad \xi J = J' \xi, \quad \xi E = E' \xi \quad \text{and} \quad g'(\xi x, \xi y) = g(x, y)$$

for all  $x, y \in \mathfrak{g}$ .

*Remark.* The first two conditions in (13) mean that the underlying complex product structures  $\{J, E\}$  and  $\{J', E'\}$  are equivalent. The third condition means that  $\xi$  is an isometry between  $g$  and  $g'$ .

**Lemma 3.8.** *With notation as in the previous definition, let  $\nabla^g$  and  $\nabla^{g'}$  be the Levi-Civita connection of  $g$  and  $g'$  respectively. Then  $\xi$  gives an equivalence between these two connections. Furthermore, if  $\omega_i, i = 1, 2, 3$ , are given as in (6) and  $\omega'_i, i = 1, 2, 3$ , are defined similarly for  $\mathfrak{g}'$ , then  $\omega_i(x, y) = \omega'_i(\xi x, \xi y)$  for all  $x, y \in \mathfrak{g}$ .*

*Proof.* The Levi-Civita connection  $\nabla^g$  of  $g$  is the only torsion-free connection on  $\mathfrak{g}$  such that  $\nabla^g J = \nabla^g E = 0$ , and a similar statement holds for  $\nabla^{g'}$ . We would like to show that  $\xi \nabla_x^g y = \nabla_{\xi x}^{g'} \xi y$  for all  $x, y \in \mathfrak{g}$ . To see this, define a connection  $\tilde{\nabla}$  on  $\mathfrak{g}$  by

$$\tilde{\nabla}_x y := \xi^{-1} \nabla_{\xi x}^{g'} \xi y, \quad x, y \in \mathfrak{g}.$$

Let us see that it is torsion-free:

$$\tilde{\nabla}_x y - \tilde{\nabla}_y x = \xi^{-1} (\nabla_{\xi x}^{g'} \xi y - \nabla_{\xi y}^{g'} \xi x) = \xi^{-1} [\xi x, \xi y] = [x, y]$$

for any  $x, y \in \mathfrak{g}$ . Let us verify now that  $J$  is  $\tilde{\nabla}$ -parallel.

$$\tilde{\nabla}_x Jy = \xi^{-1} \nabla_{\xi x}^{g'} \xi Jy = \xi^{-1} \nabla_{\xi x}^{g'} J' \xi y = \xi^{-1} J' \nabla_{\xi x}^{g'} \xi y = J \xi^{-1} \nabla_{\xi x}^{g'} \xi y = J \tilde{\nabla}_x y$$

for all  $x, y \in \mathfrak{g}$ . In the same way, it can be seen that  $\tilde{\nabla} E = 0$ . By uniqueness, we have that  $\nabla^g = \tilde{\nabla}$  and hence  $\nabla^g$  and  $\nabla^{g'}$  are equivalent.

Let us check now the assertions about the symplectic forms. Let us consider first the 2-form  $\omega_1$ . We have

$$\omega_1(x, y) = g(Jx, y) = g'(\xi Jx, \xi y) = g'(J' \xi x, \xi y) = \omega'_1(\xi x, \xi y)$$

for all  $x, y \in \mathfrak{g}$ . In a similar fashion one can prove the corresponding statements for  $\omega_2$  and  $\omega_3$ .  $\square$

Motivated by the previous result, we introduce the following definition.

**Definition 3.9.** Let  $\mathfrak{g}$  be a Lie algebra equipped with a connection  $\nabla$  and a symplectic form  $\omega$  such that  $\nabla \omega = 0$ , and similarly for a Lie algebra  $\mathfrak{g}'$  with  $\nabla'$  and  $\omega'$ . We will say that  $(\nabla, \omega)$  and  $(\nabla', \omega')$  are *symplectically equivalent* if there exists a Lie algebra isomorphism  $\xi : \mathfrak{g} \longrightarrow \mathfrak{g}'$  such that

$$\xi \nabla_x y = \nabla'_{\xi x} \xi y, \quad \omega(x, y) = \omega'(\xi x, \xi y)$$

for all  $x, y \in \mathfrak{g}$ .

**Proposition 3.10.** *Keep the notation from Theorem 3.5. Suppose that  $(\nabla, \omega)$  is symplectically equivalent to  $(\overline{\nabla}, \overline{\omega})$ , where  $\overline{\nabla}$  is a flat torsion-free connection on  $\mathfrak{u}$  and  $\overline{\omega}$  is a symplectic form on  $\mathfrak{u}$  such that  $\overline{\nabla}\overline{\omega} = 0$ . Equally, let  $(\nabla', \omega')$  be symplectically equivalent to  $(\overline{\nabla}', \overline{\omega}')$ . Then we obtain a matched pair of Lie algebras  $(\mathfrak{u}, \mathfrak{v}, \overline{\rho}, \overline{\mu})$  and the bicrossproduct  $\overline{\mathfrak{g}} = \mathfrak{u} \bowtie_{\overline{\mu}}^{\overline{\rho}} \mathfrak{v}$  has a hypersymplectic structure equivalent to the one on  $\mathfrak{g} = \mathfrak{u} \bowtie_{\mu}^{\rho} \mathfrak{v}$ .*

*Proof.* Let  $\xi : \mathfrak{u} \rightarrow \mathfrak{u}$  and  $\xi' : \mathfrak{v} \rightarrow \mathfrak{v}$  be the Lie algebra isomorphisms which give the symplectic equivalences between  $(\nabla, \omega)$  and  $(\overline{\nabla}, \overline{\omega})$  and between  $(\nabla', \omega')$  and  $(\overline{\nabla}', \overline{\omega}')$ . Consider now the linear isomorphism  $\psi : \mathfrak{u} \rightarrow \mathfrak{v}$  given by  $\psi = \xi' \varphi \xi^{-1}$ . Associated to the isomorphism  $\psi$  we have the representations  $\overline{\rho} : \mathfrak{u} \rightarrow \mathfrak{gl}(\mathfrak{v})$  and  $\overline{\mu} : \mathfrak{v} \rightarrow \mathfrak{gl}(\mathfrak{u})$  defined by

$$\overline{\rho}(x)a = \psi \overline{\nabla}_x \psi^{-1}(a), \quad \overline{\mu}(a)x = \psi^{-1} \overline{\nabla}'_a \psi(x).$$

It is easily verified that  $(\mathfrak{u}, \mathfrak{v}, \overline{\rho}, \overline{\mu})$  is a matched pair of Lie algebras, using that  $(\mathfrak{u}, \mathfrak{v}, \rho, \mu)$  is another matched pair of Lie algebras. We may form now the bicrossproduct Lie algebras  $\mathfrak{g} = \mathfrak{u} \bowtie_{\mu}^{\rho} \mathfrak{v}$  and  $\overline{\mathfrak{g}} = \mathfrak{u} \bowtie_{\overline{\mu}}^{\overline{\rho}} \mathfrak{v}$ . Furthermore, it is easy to see that  $\overline{\omega}(x, y) = \overline{\omega}'(\psi(x), \psi(y))$  for all  $x, y \in \mathfrak{u}$ . From Theorem 3.5, both  $\mathfrak{g}$  and  $\overline{\mathfrak{g}}$  have a hypersymplectic structure. Let us see now that they are equivalent. Consider the linear isomorphism  $\eta := \xi \oplus \xi' : \mathfrak{g} \rightarrow \overline{\mathfrak{g}}$  and observe that

$$\begin{aligned} [\eta(x, a), \eta(y, b)] &= [(\xi x, \xi' a), (\xi y, \xi' b)] \\ &= ([\xi x, \xi y] + \overline{\mu}(\xi' a)\xi y - \overline{\mu}(\xi' b)\xi x, [\xi' a, \xi' b] + \overline{\rho}(\xi x)\xi' b - \overline{\rho}(\xi y)\xi' a) \\ &= (\xi[x, y] + \xi(\mu(a)y) - \xi(\mu(b)x), \xi'[a, b] + \xi'(\rho(x)b) - \xi'(\rho(y)a)) \\ &= \eta([x, y] + \mu(a)y - \mu(b)x, [a, b] + \rho(x)b - \rho(y)a) \\ &= \eta[(x, a), (y, b)]. \end{aligned}$$

Thus,  $\eta$  is a Lie algebra isomorphism. Now,

$$\overline{J}\eta(x, a) = \overline{J}(\xi x, \xi' a) = (-\psi^{-1}\xi' a, \psi \xi x) = (-\xi \varphi^{-1} a, \xi' \varphi x) = \eta(-\varphi^{-1} a, \varphi x) = \eta J(x, a)$$

and

$$\overline{E}\eta(x, a) = \overline{E}(\xi x, \xi' a) = (\xi x, -\xi' a) = \eta(x, -a) = \eta E(x, a)$$

for all  $(x, a) \in \mathfrak{g}$ . Hence,  $\eta J = \overline{J}\eta$  and  $\eta E = \overline{E}\eta$ . Finally, we show that  $\eta$  is an isometry between  $g$  and  $\overline{g}$ :

$$\begin{aligned} \overline{g}(\eta(x, a), \eta(y, b)) &= \overline{g}((\xi x, \xi' a), (\xi y, \xi' b)) = \omega(\psi^{-1}\xi' b, \xi x) + \omega(\psi^{-1}\xi' a, \xi y) \\ &= \omega(\xi \varphi^{-1} b, \xi x) + \omega(\xi \varphi^{-1} a, \xi y) \\ &= \omega(\varphi^{-1} b, x) + \omega(\varphi^{-1} a, y) \\ &= g((x, a), (y, b)) \end{aligned}$$

for all  $(x, a), (y, b) \in \mathfrak{g}$ . Thus,  $\eta$  gives an equivalence between the hypersymplectic structures on  $\mathfrak{g}$  and  $\overline{\mathfrak{g}}$ .  $\square$

#### 4. SYMPLECTIC FLAT TORSION-FREE CONNECTIONS ON $\mathbb{R}^2$ AND $\mathfrak{aff}(\mathbb{R})$

In the next section, we will determine all the 4-dimensional Lie algebras which carry a hypersymplectic structure. In order to do so, we will need to know all the flat torsion-free connections that preserve a symplectic form on the 2-dimensional Lie algebras. We recall that, up to isomorphism, there are only two 2-dimensional Lie algebras, namely,  $\mathbb{R}^2$  and

the Lie algebra  $\mathfrak{aff}(\mathbb{R})$ , which has a basis  $\{e_1, e_2\}$  such that  $[e_1, e_2] = e_2$ .  $\mathfrak{aff}(\mathbb{R})$  is the Lie algebra of the Lie group  $Aff(\mathbb{R})$  of affine motions of the real line.

We start with the abelian Lie algebra  $\mathbb{R}^2$ .

**Theorem 4.1.** *Let  $\mathbb{R}^2 = \text{span}\{e_1, e_2\}$  denote the 2-dimensional abelian Lie algebra and let  $\omega = e^1 \wedge e^2$  be the canonical symplectic form on  $\mathbb{R}^2$ . Then the only non zero flat torsion-free connections  $\nabla$  on  $\mathbb{R}^2$  such that  $\nabla\omega = 0$  are the following:*

- (a)  $\nabla_{e_1}e_1 = \alpha e_2$  ( $\alpha \neq 0$ ), the other possibilities being 0;
- (b)  $\nabla_{e_2}e_2 = \alpha e_1$  ( $\alpha \neq 0$ ), the other possibilities being 0;
- (c) For  $\alpha \neq 0, \beta \neq 0$ ,

$$\begin{aligned}\nabla_{e_1}e_1 &= \alpha e_1 + \beta e_2, \\ \nabla_{e_1}e_2 &= -\frac{\alpha}{\beta}(\alpha e_1 + \beta e_2) = \nabla_{e_2}e_1, \\ \nabla_{e_2}e_2 &= \frac{\alpha^2}{\beta^2}(\alpha e_1 + \beta e_2).\end{aligned}$$

*Proof.* Let us denote

$$\begin{aligned}\nabla_{e_1}e_1 &= ae_1 + be_2, \\ \nabla_{e_1}e_2 &= ce_1 + de_2 = \nabla_{e_2}e_1, \\ \nabla_{e_2}e_2 &= ge_1 + he_2,\end{aligned}$$

with  $a, b, c, d, g, h \in \mathbb{R}$ . Since  $\nabla$  is flat, we have that  $\nabla_{e_1}\nabla_{e_2} = \nabla_{e_2}\nabla_{e_1}$ , and from this condition we obtain that

$$\begin{aligned}bg &= cd, \\ (14) \quad bc - bh + d^2 - ad &= 0, \\ ag - dg + ch - c^2 &= 0.\end{aligned}$$

Now, the condition  $\nabla\omega = 0$  holds if and only if  $\omega(\nabla_x y, z) = \omega(\nabla_x z, y)$  for all  $x, y, z \in \mathbb{R}^2$ . From this we get

$$d = -a \quad \text{and} \quad h = -c.$$

Substituting into (14), we obtain

$$\begin{aligned}(15) \quad bg &= -ac, \\ a^2 &= -bc, \\ c^2 &= ag.\end{aligned}$$

If  $a = 0$ , then  $c = 0$  and  $bg = 0$ . As  $\nabla \neq 0$ , then  $b \neq 0$  or  $g \neq 0$ . If  $b \neq 0$ , then  $g = 0$  and  $\nabla$  is of type (a) in the statement. If  $g \neq 0$ , then  $b = 0$  and  $\nabla$  is of type (b) in the statement.

Let us suppose now  $a \neq 0$ . Then  $bcg \neq 0$  and from (15) we obtain  $c = -\frac{a^2}{b}$  and  $g = \frac{a^3}{b^2}$ . Therefore,  $\nabla$  is of type (c) in the statement.  $\square$

In the next proposition we study the equivalences among the connections obtained in Theorem 4.1

**Proposition 4.2.** *Let  $\nabla$  be a non zero flat torsion-free connection on  $\mathbb{R}^2$  and  $\omega$  a  $\nabla$ -parallel symplectic form on  $\mathbb{R}^2$ . Then  $(\nabla, \omega)$  is symplectically equivalent to  $(\nabla^0, e^1 \wedge e^2)$ , where  $\{e_1, e_2\}$  is a suitable basis of  $\mathbb{R}^2$ ,  $\{e^1, e^2\}$  is the dual basis and  $\nabla^0$  is given by*

$$\nabla_{e_1}^0 e_1 = e_2, \nabla_{e_1}^0 e_2 = 0, \nabla_{e_2}^0 \equiv 0.$$

*This flat torsion-free connection on  $\mathbb{R}^2$  is complete.*

*Proof.* There exists a basis  $\{e_1, e_2\}$  of  $\mathbb{R}^2$  such that  $\omega = e^1 \wedge e^2$ . Since  $\nabla\omega = 0$ , the connection  $\nabla$  must be one of those given by Theorem 4.1.

Let us suppose first that  $\nabla$  is of type (a) in Theorem 4.1. The linear isomorphism of  $\mathbb{R}^2$  which gives the symplectic equivalence between  $\nabla$  and  $\nabla^0$  is given by

$$\xi = \begin{pmatrix} \alpha^{1/3} & 0 \\ 0 & \alpha^{-1/3} \end{pmatrix}$$

in the ordered basis  $\{e_1, e_2\}$ .

Suppose now that the connection  $\nabla$  is of type (b) in Theorem 4.1. The linear isomorphism of  $\mathbb{R}^2$  which gives the symplectic equivalence between  $\nabla$  and  $\nabla^0$  is given by

$$\xi = \begin{pmatrix} 0 & -\alpha^{1/3} \\ \alpha^{-1/3} & 0 \end{pmatrix}$$

in the ordered basis  $\{e_1, e_2\}$ .

Finally, if  $\nabla$  is of type (c) in Theorem 4.1, we may take the following isomorphism of  $\mathbb{R}^2$ :

$$\xi = \begin{pmatrix} \beta^{1/3} & -\alpha\beta^{-2/3} \\ 0 & \beta^{-1/3} \end{pmatrix}.$$

The verification of all these statements is a simple matter. We would like now to check that the connection  $\nabla^0$  is complete. In order to do so, we will use equation (1). Let  $x(t) = a_1(t)e_1 + a_2(t)e_2$  be a curve on  $\mathfrak{g}$  which satisfies  $\dot{x}(t) = -\nabla_{x(t)}^0 x(t)$ . Thus, we obtain the system of differential equations

$$\begin{cases} \dot{a}_1 = 0, \\ \dot{a}_2 = -a_1^2. \end{cases}$$

The solutions of this system are clearly defined for every  $t \in \mathbb{R}$  and therefore  $\nabla^0$  is complete.  $\square$

*Remark.* In [14], a classification of flat torsion-free connections (up to equivalence) on  $\mathbb{R}^2$  is given. The flat torsion-free connection  $\nabla^0$  from Proposition 4.2 belongs to the class  $A_4$  of that classification.

Next, we move on to consider the other 2-dimensional Lie algebra,  $\mathfrak{aff}(\mathbb{R})$ .

**Theorem 4.3.** *Let  $\mathfrak{aff}(\mathbb{R}) = \text{span}\{e_1, e_2\}$  denote the 2-dimensional Lie algebra with Lie bracket  $[e_1, e_2] = e_2$  and let  $\omega = e^1 \wedge e^2$  be the canonical symplectic form on  $\mathfrak{aff}(\mathbb{R})$ . Then the only flat torsion-free connections  $\nabla$  on  $\mathfrak{aff}(\mathbb{R})$  such that  $\nabla\omega = 0$  are the following:*

(a) For  $\alpha \in \mathbb{R}$ ,

$$\begin{aligned}\nabla_{e_1} e_1 &= -e_1 + \alpha e_2, \\ \nabla_{e_1} e_2 &= e_2, \\ \nabla_{e_2} e_1 &= \nabla_{e_2} e_2 = 0.\end{aligned}$$

(b) For  $\alpha \in \mathbb{R}$ ,

$$\begin{aligned}\nabla_{e_1} e_1 &= -\frac{1}{2}e_1 + \alpha e_2, \\ \nabla_{e_1} e_2 &= \frac{1}{2}e_2, \\ \nabla_{e_2} e_1 &= -\frac{1}{2}e_2, \\ \nabla_{e_2} e_2 &= 0.\end{aligned}$$

*Proof.* Let us denote

$$\begin{aligned}\nabla_{e_1} e_1 &= ae_1 + be_2, \\ \nabla_{e_1} e_2 &= ce_1 + de_2, \\ \nabla_{e_2} e_1 &= ge_1 + he_2,\end{aligned}$$

with  $a, b, c, d, g, h \in \mathbb{R}$ . Since  $\nabla$  is torsion-free, we have

$$\nabla_{e_2} e_1 = ce_1 + (d - 1)e_2.$$

The condition  $\nabla\omega = 0$  implies that  $d = -a$  and  $h = -c$ . Taking this into account and using that  $\nabla$  is flat, we obtain the following equations

$$\begin{aligned}(16) \quad & c(a + 2) + bg = 0, \\ & g(2a - 1) - 2c^2 = 0, \\ & 2bc + (a + 1)(2a + 1) = 0.\end{aligned}$$

From the third equation in (16) we get

$$(17) \quad 2a^2 + 3a + (2bc + 1) = 0.$$

Also, from (16) we see immediately that  $a \neq \frac{1}{2}$ . Hence  $g = \frac{2c^2}{2a-1}$  and substituting into the first equation we have

$$c \left( (a + 2) + \frac{2bc}{2a - 1} \right) = 0.$$

If  $c \neq 0$ , then  $(a + 2)(2a - 1) + 2bc = 0$  and  $2a^2 + 3a + 2bc - 2 = 0$ , which combined with (17) yields a contradiction. Thus,  $c = 0$  and the system (16) becomes

$$\begin{aligned}(18) \quad & bg = 0, \\ & g(2a - 1) = 0, \\ & (a + 1)(2a + 1) = 0.\end{aligned}$$

Therefore,  $g = 0$  (since  $a \neq \frac{1}{2}$ ),  $b \in \mathbb{R}$  is arbitrary and  $a = -1$  or  $a = -\frac{1}{2}$ . In the first case, we obtain a connection of type (a) and in the second case we obtain a connection of type (b). The proof is complete.  $\square$

In the next proposition we deal with the equivalences of the connections obtained in Theorem 4.3.

**Proposition 4.4.** *Let  $\nabla$  be a flat torsion-free connection on  $\mathfrak{aff}(\mathbb{R})$  and  $\omega$  a  $\nabla$ -parallel symplectic form on  $\mathfrak{aff}(\mathbb{R})$ . Then  $(\nabla, \omega)$  is symplectically equivalent to either  $(\nabla^1, e^1 \wedge e^2)$  or  $(\nabla^2, e^1 \wedge e^2)$ , where  $\{e_1, e_2\}$  is a suitable basis of  $\mathfrak{aff}(\mathbb{R})$ ,  $\{e^1, e^2\}$  is the dual basis and  $\nabla^1, \nabla^2$  are given by:*

$$\nabla_{e_1}^1 e_1 = -e_1, \quad \nabla_{e_1}^1 e_2 = e_2, \quad \nabla_{e_2}^1 \equiv 0$$

and

$$\nabla_{e_1}^2 e_1 = -\frac{1}{2}e_1, \quad \nabla_{e_1}^2 e_2 = \frac{1}{2}e_2, \quad \nabla_{e_2}^2 e_1 = -\frac{1}{2}e_2, \quad \nabla_{e_2}^2 e_2 = 0.$$

Both connections  $\nabla^1$  and  $\nabla^2$  on  $\mathfrak{aff}(\mathbb{R})$  are not complete.

*Proof.* Let  $\{\tilde{e}_1, \tilde{e}_2\}$  be a basis of  $\mathfrak{aff}(\mathbb{R})$  such that  $[\tilde{e}_1, \tilde{e}_2] = \tilde{e}_2$ . There exists  $\lambda \neq 0$  such that  $\omega = \lambda(\tilde{e}^1 \wedge \tilde{e}^2)$ . Set  $e_1 := \tilde{e}_1$ ,  $e_2 := \lambda\tilde{e}_2$ . We have then  $[e_1, e_2] = e_2$  and

$$\omega = \lambda(\tilde{e}^1 \wedge \tilde{e}^2) = \lambda(e^1 \wedge \lambda^{-1}e^2) = e^1 \wedge e^2.$$

So, we have  $\nabla(e^1 \wedge e^2) = 0$ , and then  $\nabla$  must be one of the flat torsion-free connections given in Theorem 4.3.

Let us suppose first that  $\nabla$  is of type (a) in Theorem 4.3. The linear isomorphism of  $\mathfrak{aff}(\mathbb{R})$  which gives the symplectic equivalence between  $\nabla$  and  $\nabla^1$  is given by

$$\xi = \begin{pmatrix} 1 & 0 \\ \frac{1}{2}\alpha & 1 \end{pmatrix}$$

in the ordered basis  $\{e_1, e_2\}$ .

If we take now a connection  $\nabla$  of type (b) in Theorem 4.3, the linear isomorphism of  $\mathfrak{aff}(\mathbb{R})$  giving the symplectic equivalence between  $\nabla$  and  $\nabla^2$  is

$$\xi = \begin{pmatrix} 1 & 0 \\ 2\alpha & 1 \end{pmatrix}$$

in the ordered basis  $\{e_1, e_2\}$ .

Next, we observe that  $\nabla^1$  and  $\nabla^2$  are not equivalent. If they were, the subspaces  $W_1 = \{x \in \mathfrak{aff}(\mathbb{R}) : \nabla_x^1 \equiv 0\}$  and  $W_2 = \{x \in \mathfrak{aff}(\mathbb{R}) : \nabla_x^2 \equiv 0\}$  of  $\mathfrak{aff}(\mathbb{R})$  should be isomorphic. However, it is clear that  $\dim W_1 = 1$  while  $W_2 = \{0\}$ . Thus, these two connections are not equivalent.

Finally, we show that these connections are not complete. Suppose  $x(t) = a_1(t)e_1 + a_2(t)e_2$  is a curve on  $\mathfrak{aff}(\mathbb{R})$  that satisfies  $\dot{x}(t) = -\nabla_{x(t)}^1 x(t)$ . Thus, we obtain the system of differential equations

$$\begin{cases} \dot{a}_1 = a_1^2, \\ \dot{a}_2 = -a_1 a_2. \end{cases}$$

From the first equation in the system we obtain that  $a_1(t)$  cannot be defined in the whole real line; thus  $\nabla^1$  is not complete. Analogously, if  $x(t) = a_1(t)e_1 + a_2(t)e_2$  is a curve on  $\mathfrak{aff}(\mathbb{R})$  that satisfies  $\dot{x}(t) = -\nabla_{x(t)}^2 x(t)$ , we have the system

$$\begin{cases} \dot{a}_1 = \frac{1}{2}a_1^2, \\ \dot{a}_2 = 0. \end{cases}$$

We obtain again that  $a_1(t)$  cannot be defined in the whole real line; thus  $\nabla^2$  is not complete.  $\square$

## 5. HYPERSYMPLECTIC 4-DIMENSIONAL LIE ALGEBRAS

In this section we will determine all 4-dimensional Lie algebras which carry a hypersymplectic structure, by employing Theorem 3.5. We will also be able to obtain a parametrization of the underlying complex product structures, up to equivalence. In the next section we will exhibit explicit descriptions of the hypersymplectic metrics in each case.

A classification of 4-dimensional Lie algebras admitting a complex product structure was given by Blazić and Vukmirović in [4], where they refer to complex product structures as para-hypercomplex structures. The family of Lie algebras that we will obtain below can be found within this classification.

We introduce first some notation. We will consider the following 4-dimensional Lie algebras:

- $\mathfrak{g}_0^h = \text{span}\{v_0, v_1, v_2, v_3\}$  with  $[v_1, v_2] = v_3$ ,
- $\mathfrak{g}_1^h = \text{span}\{v_0, v_1, v_2, v_3\}$  with  $[v_0, v_1] = v_1$ ,  $[v_0, v_2] = -v_2$ ,  $[v_0, v_3] = -v_3$ ,
- $\mathfrak{g}_2^h = \text{span}\{v_0, v_1, v_2, v_3\}$  with  $[v_0, v_1] = 2v_1$ ,  $[v_0, v_2] = -v_2$ ,  $[v_0, v_3] = v_3$ ,  $[v_1, v_2] = v_3$ .

*Remarks.* (i) The Lie algebra  $\mathfrak{g}_0^h \cong \mathfrak{h}_3 \times \mathbb{R}$  is a central extension of the 3-dimensional Heisenberg algebra. It is the only 2-step nilpotent 4-dimensional Lie algebra.

(ii) The Lie algebra  $\mathfrak{g}_1^h$  is an extension of  $\mathbb{R}^3$  and it lies in the class  $\mathfrak{r}_{4,-1,-1}$  of the classification of 4-dimensional solvable Lie algebras given in [1]. It is 2-step solvable and not unimodular.

(iii) The Lie algebra  $\mathfrak{g}_2^h$  is an extension of  $\mathfrak{h}_3$  and it lies in the class  $\mathfrak{d}_{4,2}$  of the classification of 4-dimensional solvable Lie algebras given in [1]. It is 3-step solvable and not unimodular.

**Theorem 5.1.** *Let  $\mathfrak{g}$  be a 4-dimensional Lie algebra carrying a hypersymplectic structure. Then  $\mathfrak{g}$  is isomorphic to either  $\mathbb{R}^4$ ,  $\mathfrak{g}_0^h$ ,  $\mathfrak{g}_1^h$  or  $\mathfrak{g}_2^h$ . Furthermore, the parametrization of the complex product structures in each case is given by:*

- (i) *If  $\mathfrak{g} \cong \mathbb{R}^4$ , the underlying complex product structure is equivalent to  $\{J, E\}$ , where*

$$J = \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \quad E = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$$

*(with  $\mathbf{1}$  the  $(2 \times 2)$ -identity matrix) in some ordered basis of  $\mathbb{R}^4$ .*

- (ii) *If  $\mathfrak{g} \cong \mathfrak{g}_0^h$ , then the underlying complex product structure on  $\mathfrak{g}$  is equivalent to one and only one of  $\{J^{(0)}, E_\theta^{(0)}\}$ , where*

$$J^{(0)} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad E_\theta^{(0)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \cos \theta & \sin \theta \\ 0 & 0 & \sin \theta & -\cos \theta \end{pmatrix}$$

*for  $\theta \in [0, 2\pi)$ , in the ordered basis  $\{v_1, v_2, v_3, v_0\}$ .*



(iii) If  $\mathfrak{g} \cong \mathfrak{g}_1^h$ , then the underlying complex product structure on  $\mathfrak{g}$  is equivalent to one and

only one of  $\{J^{(1)}, E_{\theta,d}^{(1)}\}$  or  $\{J^{(1)}, E_1^{(1)}\}$ , where  $J^{(1)} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$  and

$$E_{\theta,d}^{(1)} = \begin{pmatrix} \cos \theta & \sin \theta & 0 & 0 \\ \sin \theta & -\cos \theta & 0 & 0 \\ -d \sin \theta & d(1 + \cos \theta) & 1 & 0 \\ d(1 + \cos \theta) & d \sin \theta & 0 & -1 \end{pmatrix}, \quad E_1^{(1)} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 2 & 0 & -1 \end{pmatrix}$$

for  $\theta \in [0, 2\pi)$  and  $d = 0$  or  $d = 1$ , in the ordered basis  $\{v_0, v_1, v_2, v_3\}$ .

(iv) If  $\mathfrak{g} \cong \mathfrak{g}_2^h$ , then the underlying complex product structure on  $\mathfrak{g}$  is equivalent to one and

only one of  $\{J^{(2)}, E_{\theta,d}^{(2)}\}$  or  $\{J^{(2)}, E_1^{(2)}\}$ , where  $J^{(2)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$  and

$$E_{\theta,d}^{(2)} = \begin{pmatrix} \cos \theta & \sin \theta & 0 & 0 \\ \sin \theta & -\cos \theta & 0 & 0 \\ d \sin \theta(1 + \cos \theta) & -d \cos \theta(1 + \cos \theta) & \cos \theta & -\sin \theta \\ d \cos \theta(1 + \cos \theta) & d \sin \theta(1 + \cos \theta) & -\sin \theta & -\cos \theta \end{pmatrix}, \quad E_1^{(2)} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ -2 & 0 & 0 & 1 \end{pmatrix}$$

for  $\theta \in [0, 2\pi)$  and  $d = 0$  or  $d = 1$ , in the ordered basis  $\{v_0, v_2, v_1, v_3\}$ .

*Remark.* Note that  $E_{\pi,0}^{(1)} = E_{\pi,1}^{(1)}$  and  $E_{\pi,0}^{(2)} = E_{\pi,1}^{(2)}$ .

*Proof.* We will construct explicitly all 4-dimensional Lie algebras carrying a hypersymplectic structure using Theorem 3.5. In order to do so, we have to determine the two 3-tuples  $(\mathfrak{u}, \nabla, \omega)$ ,  $(\mathfrak{v}, \nabla', \omega')$  and the linear isomorphism  $\varphi : \mathfrak{u} \longrightarrow \mathfrak{v}$  which satisfy the conditions of this theorem. The Lie algebras  $\mathfrak{u}$  and  $\mathfrak{v}$  are 2-dimensional, and therefore they are isomorphic either to  $\mathbb{R}^2$  or  $\mathfrak{aff}(\mathbb{R})$ . The flat torsion-free connections on these Lie algebras which are compatible with the canonical symplectic forms were determined in §4. We only have to establish the linear isomorphisms  $\varphi$  which are admissible. We will do this in several steps.

**Case (A):  $\mathfrak{u} = \mathbb{R}^2$  and  $\mathfrak{v} = \mathbb{R}^2$ .**

We fix a basis  $\{e_1, e_2\}$  of  $\mathfrak{u}$  and its associated symplectic form  $\omega = e^1 \wedge e^2$ , where  $\{e^1, e^2\}$  is the dual basis. In the same way we fix a basis  $\{f_1, f_2\}$  of  $\mathfrak{v}$  and its associated symplectic form  $\omega' = f^1 \wedge f^2$ , where  $\{f^1, f^2\}$  is the dual basis. In this case there are only two connections to be considered: the connection identically zero and the connection  $\nabla^0$  which appears in Proposition 4.2.

(A1)  $\nabla = 0$  and  $\nabla' = 0$ .

Here  $\mathfrak{g} = \mathfrak{u} \ltimes \mathfrak{v} = \mathbb{R}^4$  is the abelian 4-dimensional Lie algebra and the complex product structure is the canonical one, given by the matrices in the statement of the theorem.

(A2)  $\nabla = \nabla^0$  and  $\nabla' = 0$ .

Here  $\mathfrak{g} = \mathfrak{u} \ltimes \mathfrak{v} = \mathbb{R}^2 \ltimes \mathbb{R}^2$ . In this special case we may simply suppose that the linear isomorphism  $\varphi : \mathfrak{aff}(\mathbb{R}) \longrightarrow \mathbb{R}^2$  that we are seeking satisfies  $\varphi(e_i) = f_i$ ,  $i = 1, 2$ . It is easy to see that this isomorphism is compatible with  $\nabla$  and  $\nabla'$  and also with  $\omega$  and  $\omega'$ . Hence,

we obtain a hypersymplectic structure on  $\mathfrak{g}$ . Let us identify this Lie algebra. If we denote  $e_i := (e_i, 0)$  and  $f_i := (0, f_i)$  for  $i = 1, 2$ , then the only non-zero bracket is  $[e_1, f_1] = f_2$ . We also have  $Je_i = f_i$  and  $Ee_i = e_i$ ,  $Ef_i = -f'_i$ . We will make now a change of basis, setting:

$$v_1 := e_1, \quad v_2 := f_1, \quad v_3 := f_2, \quad v_0 := -e_2.$$

Then we have  $[v_1, v_2] = v_3$  and therefore  $\mathfrak{g} \cong \mathfrak{g}_0^h$ . The complex structure  $J$  is given by  $Jv_1 = v_2$ ,  $Jv_3 = v_0$  and the eigenspaces corresponding to  $E$  are  $\mathfrak{g}_+ = \text{span}\{v_0, v_1\}$  and  $\mathfrak{g}_- = \text{span}\{v_2, v_3\}$ . This complex product structure is equivalent to  $\{J^{(0)}, E_\theta^{(0)}\}$  with  $\theta = \pi$ .

(A3)  $\nabla = 0$  and  $\nabla' = \nabla^0$ .

Here  $\mathfrak{g} = \mathfrak{u} \bowtie \mathfrak{v} = \mathbb{R}^2 \ltimes \mathbb{R}^2$ . In this special case we may simply suppose that the linear isomorphism  $\varphi : \mathfrak{aff}(\mathbb{R}) \rightarrow \mathbb{R}^2$  that we are seeking satisfies  $\varphi(e_i) = f_i$ ,  $i = 1, 2$ . It is easy to see that this isomorphism is compatible with  $\nabla$  and  $\nabla'$  and also with  $\omega$  and  $\omega'$ . Hence, we obtain a hypersymplectic structure on  $\mathfrak{g}$ . Let us identify this Lie algebra. If we denote  $e_i := (e_i, 0)$  and  $f_i := (0, f_i)$  for  $i = 1, 2$ , then the only non zero bracket is  $[e_1, f_1] = e_2$ . We have that  $Je_i = f_i$  and  $Ee_i = e_i$ ,  $Ef_i = -f_i$ . We will make now a change of basis, setting:

$$v_1 := e_1, \quad v_2 := f_1, \quad v_3 := e_2, \quad v_0 := f_2.$$

Then we have  $[v_1, v_2] = v_3$  and therefore  $\mathfrak{g} \cong \mathfrak{g}_0^h$ . The complex structure  $J$  is given by  $Jv_1 = v_2$ ,  $Jv_3 = v_0$  and the eigenspaces corresponding to  $E$  are  $\mathfrak{g}_+ = \text{span}\{v_1, v_3\}$  and  $\mathfrak{g}_- = \text{span}\{v_0, v_2\}$ . This complex product structure is equivalent to  $\{J^{(0)}, E_\theta^{(0)}\}$  with  $\theta = 0$ .

(A4)  $\nabla = \nabla^0$  and  $\nabla' = \nabla^0$ .

We are looking for a linear isomorphism  $\varphi : \mathfrak{u} \rightarrow \mathfrak{v}$  compatible with  $\nabla$  and  $\nabla'$  and also with  $\omega$  and  $\omega'$ . After lengthy computations, we obtain that  $\varphi$  must be of the form  $\varphi = \begin{pmatrix} a & 0 \\ b & d \end{pmatrix}$ , with  $ad = 1$  (in the ordered bases  $\{e_1, e_2\}, \{f_1, f_2\}$ ). Hence, we have a hypersymplectic structure on the bicrossproduct Lie algebra  $\mathfrak{g} := \mathfrak{u} \bowtie \mathfrak{v} = \mathbb{R}^2 \bowtie \mathbb{R}^2$ . Let us describe this Lie algebra. Let us denote  $e_i := (e_i, 0)$ ,  $f_i := (0, f_i)$  for  $i = 1, 2$ ; the only non zero bracket is  $[e_1, f_1] = -a^2e_2 + d^2f_2$ . The complex product structure on this Lie algebra is given by

$$Je_1 = af_1 + bf_2, \quad Je_2 = df_2, \quad Jf_1 = -de_1 + be_2, \quad Jf_2 = -ae_2$$

and  $Ee_i = e_i$ ,  $Ef_i = -f_i$  for  $i = 1, 2$ . We will make now a change of basis, setting:

$$v_1 := e_1, \quad v_2 := af_1 + bf_2, \quad v_3 := a(-a^2e_2 + d^2f_2), \quad v_0 := -a(de_2 + af_2).$$

Then we have  $[v_1, v_2] = v_3$  and hence  $\mathfrak{g} \cong \mathfrak{g}_0^h$ . The complex structure  $J$  is given by  $Jv_1 = v_2$ ,  $Jv_3 = v_0$  and the eigenspaces corresponding to  $E$  are

$$\mathfrak{g}_+ = \text{span} \left\{ v_1, \frac{a^3}{a^6+1}v_3 + \frac{1}{a^6+1}v_0 \right\}, \quad \mathfrak{g}_- = \text{span} \left\{ v_2, -\frac{1}{a^6+1}v_3 + \frac{a^3}{a^6+1}v_0 \right\}.$$

This complex product structure is equivalent to  $\{J^{(0)}, E_\theta^{(0)}\}$ , where  $\theta$  is given by  $\cos(\theta/2) = \frac{a^3}{\sqrt{a^6+1}}$ ,  $\sin(\theta/2) = \frac{1}{\sqrt{a^6+1}}$ . Note that  $\theta \neq 0$  and  $\theta \neq \pi$ .

**Case (B):**  $\mathfrak{u} = \mathfrak{aff}(\mathbb{R})$  and  $\mathfrak{v} = \mathbb{R}^2$ .

We fix a basis  $\{e_1, e_2\}$  of  $\mathfrak{u}$  such that  $[e_1, e_2] = e_2$  and its associated symplectic form  $\omega = e^1 \wedge e^2$ , where  $\{e^1, e^2\}$  is the dual basis. In the same way we fix a basis  $\{f_1, f_2\}$  of  $\mathfrak{v}$  and its associated symplectic form  $\omega' = f^1 \wedge f^2$ , where  $\{f^1, f^2\}$  is the dual basis.

(B1)  $\nabla = \nabla^1$  and  $\nabla' = 0$ .

In this case, we have  $\mathfrak{g} := \mathfrak{u} \bowtie \mathfrak{v} = \mathfrak{aff}(\mathbb{R}) \ltimes \mathbb{R}^2$ . Let us describe this Lie algebra. If we denote  $e_i := (e_i, 0)$ ,  $f_i := (0, f_i)$  for  $i = 1, 2$ , then the only non zero brackets are

$$[e_1, e_2] = e_2, \quad [e_1, f_1] = -f_1, \quad [e_1, f_2] = f_2.$$

The complex product structure on this Lie algebra is given by  $Je_i = f_i$  and  $Ee_i = e_i$ ,  $Ef_i = -f_i$  for  $i = 1, 2$ . We will make a change of basis, setting  $v_0 = -e_1$ ,  $v_1 = -f_1$ ,  $v_2 = e_2$ ,  $v_3 = f_2$ . Thus,

$$\begin{aligned} [v_0, v_1] &= v_1, & [v_0, v_2] &= -v_2, & [v_0, v_3] &= -v_3, \\ Jv_0 &= v_1, & Jv_2 &= v_3, \end{aligned}$$

and then  $\mathfrak{g} \cong \mathfrak{g}_1^h$ . The eigenspaces corresponding to  $E$  are

$$\mathfrak{g}_+ = \text{span}\{v_0, v_2\}, \quad \mathfrak{g}_- = \text{span}\{v_1, v_3\}.$$

This complex product structure is equivalent to  $\{J^{(1)}, E_{\theta,0}^{(1)}\}$  with  $\theta = 0$ .

(B2)  $\nabla = \nabla^1$  and  $\nabla' = \nabla^0$ .

We are seeking a linear isomorphism  $\varphi : \mathfrak{aff}(\mathbb{R}) \longrightarrow \mathbb{R}^2$  compatible with  $\nabla$  and  $\nabla'$  and also with  $\omega$  and  $\omega'$ . It can be seen that  $\varphi$  must be of the form  $\varphi = \begin{pmatrix} a & 0 \\ b & d \end{pmatrix}$ , with  $ad = 1$  (in the ordered bases  $\{e_1, e_2\}$ ,  $\{f_1, f_2\}$ ). Hence, we have a hypersymplectic structure on the bicrossproduct Lie algebra  $\mathfrak{g} := \mathfrak{u} \bowtie \mathfrak{v} = \mathfrak{aff}(\mathbb{R}) \bowtie \mathbb{R}^2$ . Let us denote  $e_i := (e_i, 0)$ ,  $f_i := (0, f_i)$  for  $i = 1, 2$ ; the only non zero brackets are

$$[e_1, e_2] = e_2, \quad [e_1, f_1] = -a^2 e_2 - f_1 - 2bdf_2, \quad [e_1, f_2] = f_2.$$

The complex product structure on this Lie algebra is given by

$$Je_1 = af_1 + bf_2, \quad Je_2 = df_2, \quad Jf_1 = -de_1 + be_2, \quad Jf_2 = -ae_2$$

and  $Ee_i = e_i$ ,  $Ef_i = -f_i$  for  $i = 1, 2$ . We will make now a change of basis, setting  $v_0 := -e_1 + \frac{a^2}{2}f_2$ ,  $v_1 := -\frac{a^3}{2}e_2 - af_1 - bf_2$ ,  $v_2 := e_2$ ,  $v_3 := df_2$ . Thus,

$$\begin{aligned} [v_0, v_1] &= v_1, & [v_0, v_2] &= -v_2, & [v_0, v_3] &= -v_3, \\ Jv_0 &= v_1, & Jv_2 &= v_3, \end{aligned}$$

and then  $\mathfrak{g} \cong \mathfrak{g}_1^h$ . The eigenspaces corresponding to  $E$ , in this new basis, are given by

$$\mathfrak{g}_+ = \text{span}\left\{v_0 - \frac{a^3}{2}v_3, v_2\right\}, \quad \mathfrak{g}_- = \text{span}\left\{v_1 + \frac{a^3}{2}v_2, v_3\right\}.$$

This complex product structure is equivalent to  $\{J^{(1)}, E_{\theta,1}^{(1)}\}$  with  $\theta = 0$ .

(B3)  $\nabla = \nabla^2$  and  $\nabla' = 0$ .

In this case, we have  $\mathfrak{g} := \mathfrak{u} \bowtie \mathfrak{v} = \mathfrak{aff}(\mathbb{R}) \ltimes \mathbb{R}^2$ . Let us describe this Lie algebra. If we denote  $e_i := (e_i, 0)$ ,  $f_i := (0, f_i)$  for  $i = 1, 2$ , then the only non zero brackets are

$$[e_1, e_2] = e_2, \quad [e_1, f_1] = -\frac{1}{2}f_1, \quad [e_1, f_2] = \frac{1}{2}f_2, \quad [e_2, f_1] = -\frac{1}{2}f_2.$$

The complex product structure on this Lie algebra is given by  $Je_i = f_i$  and  $Ee_i = e_i$ ,  $Ef_i = -f_i$  for  $i = 1, 2$ . We will make a change of basis, setting  $v_0 := 2e_1$ ,  $v_1 := e_2$ ,  $v_2 := -2f_1$ ,  $v_3 := f_2$ . Thus,

$$\begin{aligned} [v_0, v_1] &= 2v_1, & [v_0, v_2] &= -v_2, & [v_0, v_3] &= v_3, & [v_1, v_2] &= v_3, \\ Jv_0 &= -v_2, & Jv_1 &= v_3, \end{aligned}$$

and then  $\mathfrak{g} \cong \mathfrak{g}_2^h$ . The eigenspaces corresponding to  $E$ , in this new basis, are given by

$$\mathfrak{g}_+ = \text{span}\{v_0, v_1\}, \quad \mathfrak{g}_- = \text{span}\{v_2, v_3\}.$$

This complex product structure is equivalent to  $\{J^{(2)}, E_{\theta,0}^{(2)}\}$  with  $\theta = 0$ .

(B4)  $\nabla = \nabla^2$  and  $\nabla' = \nabla^0$ .

We are looking for a linear isomorphism  $\varphi : \mathfrak{aff}(\mathbb{R}) \longrightarrow \mathbb{R}^2$  compatible with  $\nabla$  and  $\nabla'$  and also with  $\omega$  and  $\omega'$ . Again,  $\varphi$  must be of the form  $\varphi = \begin{pmatrix} a & 0 \\ b & d \end{pmatrix}$ , with  $ad = 1$  (in the ordered bases  $\{e_1, e_2\}$ ,  $\{f_1, f_2\}$ ). Hence, we have a hypersymplectic structure on the bicrossproduct Lie algebra  $\mathfrak{g} := \mathfrak{u} \bowtie \mathfrak{v} = \mathfrak{aff}(\mathbb{R}) \bowtie \mathbb{R}^2$ . Let us denote  $e_i := (e_i, 0)$ ,  $f_i := (0, f_i)$  for  $i = 1, 2$ ; the only non zero brackets are

$$[e_1, e_2] = e_2, \quad [e_1, f_1] = -a^2 e_2 - \frac{1}{2} f_1 - bdf_2, \quad [e_1, f_2] = \frac{1}{2} f_2, \quad [e_2, f_1] = -\frac{1}{2} d^2 f_2.$$

The complex product structure on this Lie algebra is given by

$$Je_1 = af_1 + bf_2, \quad Je_2 = df_2, \quad Jf_1 = -de_1 + be_2, \quad Jf_2 = -ae_2$$

and  $Ee_i = e_i$ ,  $Ef_i = -f_i$  for  $i = 1, 2$ . We will make now a change of basis, setting  $v_0 := 2e_1 - \frac{4}{3}a^2 f_2$ ,  $v_1 := e_2$ ,  $v_2 := -(\frac{4}{3}a^3 e_2 + 2af_1 + 2bf_2)$ ,  $v_3 := df_2$ . Thus,

$$\begin{aligned} [v_0, v_1] &= 2v_1, & [v_0, v_2] &= -v_2, & [v_0, v_3] &= v_3, & [v_1, v_2] &= v_3, \\ Jv_0 &= -v_2, & Jv_1 &= v_3, \end{aligned}$$

and then  $\mathfrak{g} \cong \mathfrak{g}_2^h$ . The eigenspaces corresponding to  $E$ , in this new basis, are given by

$$\mathfrak{g}_+ = \text{span}\left\{v_0 + \frac{4}{3}a^3 v_3, v_1\right\}, \quad \mathfrak{g}_- = \text{span}\left\{v_2 + \frac{4}{3}a^3 v_1, v_3\right\}.$$

This complex product structure is equivalent to  $\{J^{(2)}, E_{\theta,1}^{(2)}\}$  with  $\theta = 0$ .

**Case (B'):**  $\mathfrak{u} = \mathbb{R}^2$  and  $\mathfrak{v} = \mathfrak{aff}(\mathbb{R})$ .

We fix a basis  $\{e_1, e_2\}$  of  $\mathfrak{u}$  and its associated symplectic form  $\omega = e^1 \wedge e^2$ , where  $\{e^1, e^2\}$  is the dual basis. In the same way we fix a basis  $\{f_1, f_2\}$  of  $\mathfrak{v}$  such that  $[f_1, f_2] = f_2$  and its associated symplectic form  $\omega' = f^1 \wedge f^2$ , where  $\{f^1, f^2\}$  is the dual basis.

(B1')  $\nabla = 0$  and  $\nabla' = \nabla^1$ .

In this case, we have  $\mathfrak{g} := \mathfrak{u} \bowtie \mathfrak{v} = \mathfrak{aff}(\mathbb{R}) \bowtie \mathbb{R}^2$ . If we denote  $e_i := (e_i, 0)$ ,  $f_i := (0, f_i)$  for  $i = 1, 2$ , then the only non zero brackets are

$$[e_1, f_1] = e_1, \quad [e_2, f_1] = -e_2, \quad [f_1, f_2] = f_2.$$

The complex product structure on this Lie algebra is given by  $Je_i = f_i$  and  $Ee_i = e_i$ ,  $Ef_i = -f_i$  for  $i = 1, 2$ . We will make a change of basis, setting  $v_0 = -f_1$ ,  $v_1 = e_1$ ,  $v_2 = e_2$ ,  $v_3 = f_2$ . Thus,

$$[v_0, v_1] = v_1, \quad [v_0, v_2] = -v_2, \quad [v_0, v_3] = -v_3,$$

$$Jv_0 = v_1, \quad Jv_2 = v_3,$$

and then  $\mathfrak{g} \cong \mathfrak{g}_1^h$ . The eigenspaces corresponding to  $E$  are

$$\mathfrak{g}_+ = \text{span}\{v_1, v_2\}, \quad \mathfrak{g}_- = \text{span}\{v_0, v_3\}.$$

This complex product structure is equivalent to  $\{J^{(1)}, E_{\theta,0}^{(1)} = E_{\theta,1}^{(1)}\}$  with  $\theta = \pi$ .

(B2')  $\nabla = \nabla^0$  and  $\nabla' = \nabla^1$ .

We are seeking a linear isomorphism  $\varphi : \mathfrak{aff}(\mathbb{R}) \longrightarrow \mathbb{R}^2$  compatible with  $\nabla$  and  $\nabla'$  and also with  $\omega$  and  $\omega'$ . It can be seen that  $\varphi$  must be of the form  $\varphi = \begin{pmatrix} a & 0 \\ b & d \end{pmatrix}$ , with  $ad = 1$  (in the ordered bases  $\{e_1, e_2\}, \{f_1, f_2\}$ ). Hence, we have a hypersymplectic structure on the bicrossproduct Lie algebra  $\mathfrak{g} := \mathfrak{u} \bowtie \mathfrak{v} = \mathbb{R}^2 \bowtie \mathfrak{aff}(\mathbb{R})$ . Let us denote  $e_i := (e_i, 0)$ ,  $f_i := (0, f_i)$  for  $i = 1, 2$ ; the only non zero brackets are

$$[e_1, f_1] = e_1 - 2abe_2 + d^2f_2, \quad [e_2, f_1] = -e_2, \quad [f_1, f_2] = f_2.$$

The complex product structure on this Lie algebra is given by

$$Je_1 = af_1 + bf_2, \quad Je_2 = df_2, \quad Jf_1 = -de_1 + be_2, \quad Jf_2 = -ae_2$$

and  $Ee_i = e_i$ ,  $Ef_i = -f_i$  for  $i = 1, 2$ . We will make now a change of basis, setting  $v_0 := \frac{d^2}{2}e_2 - f_1$ ,  $v_1 := de_1 - be_2 + \frac{d^3}{2}f_2$ ,  $v_2 := e_2$ ,  $v_3 := df_2$ . Thus,

$$[v_0, v_1] = v_1, \quad [v_0, v_2] = -v_2, \quad [v_0, v_3] = -v_3,$$

$$Jv_0 = v_1, \quad Jv_2 = v_3,$$

and then  $\mathfrak{g} \cong \mathfrak{g}_1^h$ . The eigenspaces corresponding to  $E$ , in this new basis, are given by

$$\mathfrak{g}_+ = \text{span}\left\{v_1 - \frac{d^2}{2}v_3, v_2\right\}, \quad \mathfrak{g}_- = \text{span}\left\{-v_0 + \frac{d^2}{2}v_2, v_3\right\}.$$

This complex product structure is equivalent to  $\{J^{(1)}, E_1^{(1)}\}$ .

(B3')  $\nabla = 0$  and  $\nabla' = \nabla^2$ .

In this case, we have  $\mathfrak{g} := \mathfrak{u} \bowtie \mathfrak{v} = \mathfrak{aff}(\mathbb{R}) \ltimes \mathbb{R}^2$ . If we denote  $e_i := (e_i, 0)$ ,  $f_i := (0, f_i)$  for  $i = 1, 2$ , then the only non zero brackets are

$$[e_1, f_1] = \frac{1}{2}e_1, \quad [e_1, f_2] = \frac{1}{2}e_2, \quad [e_2, f_1] = -\frac{1}{2}e_2, \quad [f_1, f_2] = f_2.$$

The complex product structure on this Lie algebra is given by  $Je_i = f_i$  and  $Ee_i = e_i$ ,  $Ef_i = -f_i$  for  $i = 1, 2$ . We will make a change of basis, setting  $v_0 := 2f_1$ ,  $v_1 := f_2$ ,  $v_2 := 2e_1$ ,  $v_3 := -e_2$ . Thus,

$$[v_0, v_1] = 2v_1, \quad [v_0, v_2] = -v_2, \quad [v_0, v_3] = v_3, \quad [v_1, v_2] = v_3,$$

$$Jv_0 = -v_2, \quad Jv_1 = v_3,$$

and then  $\mathfrak{g} \cong \mathfrak{g}_2^h$ . The eigenspaces corresponding to  $E$ , in this new basis, are given by

$$\mathfrak{g}_+ = \text{span}\{v_2, v_3\}, \quad \mathfrak{g}_- = \text{span}\{v_0, v_1\}.$$

This complex product structure is equivalent to  $\{J^{(2)}, E_{\theta,0}^{(2)} = E_{\theta,1}^{(2)}\}$  with  $\theta = \pi$ .

(B4')  $\nabla = \nabla^0$  and  $\nabla' = \nabla^2$ .

We are looking for a linear isomorphism  $\varphi : \mathfrak{aff}(\mathbb{R}) \longrightarrow \mathbb{R}^2$  compatible with  $\nabla$  and  $\nabla'$  and also with  $\omega$  and  $\omega'$ . Again,  $\varphi$  must be of the form  $\varphi = \begin{pmatrix} a & 0 \\ b & d \end{pmatrix}$ , with  $ad = 1$  (in the ordered bases  $\{e_1, e_2\}, \{f_1, f_2\}$ ). Hence, we have a hypersymplectic structure on the bicrossproduct Lie algebra  $\mathfrak{g} := \mathfrak{u} \bowtie \mathfrak{v} = \mathbb{R}^2 \bowtie \mathfrak{aff}(\mathbb{R})$ . Let us denote  $e_i := (e_i, 0)$ ,  $f_i := (0, f_i)$  for  $i = 1, 2$ ; the only non zero brackets are

$$[e_1, f_1] = \frac{1}{2}e_1 - abe_2 + d^2f_2, \quad [e_1, f_2] = \frac{1}{2}a^2e_2, \quad [e_2, f_1] = -\frac{1}{2}e_2, \quad [f_1, f_2] = f_2.$$

The complex product structure on this Lie algebra is given by

$$Je_1 = af_1 + bf_2, \quad Je_2 = df_2, \quad Jf_1 = -de_1 + be_2, \quad Jf_2 = -ae_2$$

and  $Ee_i = e_i$ ,  $Ef_i = -f_i$  for  $i = 1, 2$ . We will make now a change of basis, setting  $v_0 := -\frac{4}{3}d^2e_2 + 2f_1$ ,  $v_1 := f_2$ ,  $v_2 := 2de_1 - 2be_2 + \frac{4}{3}d^3f_2$ ,  $v_3 := -ae_2$ . Thus,

$$[v_0, v_1] = 2v_1, \quad [v_0, v_2] = -v_2, \quad [v_0, v_3] = v_3, \quad [v_1, v_2] = v_3,$$

$$Jv_0 = -v_2, \quad Jv_1 = v_3,$$

and then  $\mathfrak{g} \cong \mathfrak{g}_2^h$ . The eigenspaces corresponding to  $E$ , in this new basis, are given by

$$\mathfrak{g}_+ = \text{span} \left\{ -v_2 + \frac{4}{3}d^3v_1, v_3 \right\}, \quad \mathfrak{g}_- = \text{span} \left\{ v_0 - \frac{4}{3}d^3v_3, v_1 \right\}.$$

This complex product structure is equivalent to  $\{J^{(2)}, E_1^{(2)}\}$ .

**Case (C):  $\mathfrak{u} = \mathfrak{aff}(\mathbb{R})$  and  $\mathfrak{v} = \mathfrak{aff}(\mathbb{R})$ .**

We will use  $\mathfrak{u} = \text{span}\{e_1, e_2\}$  with  $[e_1, e_2] = e_2$  and  $\mathfrak{v} = \text{span}\{f_1, f_2\}$  with  $[f_1, f_2] = f_2$ ; the symplectic forms are  $\omega = e^1 \wedge e^2$  and  $\omega' = f^1 \wedge f^2$ . Clearly, none of the connections on  $\mathfrak{u}$  or  $\mathfrak{v}$  may be zero, since in that case the Lie algebra would turn out to be abelian.

(C1)  $\nabla = \nabla^1$  and  $\nabla' = \nabla^1$ .

We are looking for a linear isomorphism  $\varphi : \mathfrak{u} \longrightarrow \mathfrak{v}$  compatible with  $\nabla$  and  $\nabla'$  and also with  $\omega$  and  $\omega'$ . After lengthy computations, we obtain that  $\varphi$  must be of the form  $\varphi = \begin{pmatrix} a & 0 \\ b & d \end{pmatrix}$ , with  $ad = 1$  (in the ordered bases  $\{e_1, e_2\}, \{f_1, f_2\}$ ). Hence, we have a hypersymplectic structure on the bicrossproduct Lie algebra  $\mathfrak{g} := \mathfrak{u} \bowtie \mathfrak{v} = \mathfrak{aff}(\mathbb{R}) \bowtie \mathfrak{aff}(\mathbb{R})$ . Let us describe this Lie algebra. Let us denote  $e_i := (e_i, 0)$ ,  $f_i := (0, f_i)$  for  $i = 1, 2$ ; the only non zero brackets are

$$[e_1, e_2] = e_2, \quad [e_1, f_1] = e_1 - 2abe_2 - f_1 - 2bdf_2, \\ [e_1, f_2] = f_2, \quad [e_2, f_1] = -e_2, \quad [f_1, f_2] = f_2.$$

The complex product structure on this Lie algebra is given by

$$Je_1 = af_1 + bf_2, \quad Je_2 = df_2, \quad Jf_1 = -de_1 + be_2, \quad Jf_2 = -ae_2$$

and  $Ee_i = e_i$ ,  $Ef_i = -f_i$  for  $i = 1, 2$ . We will make a change of basis, setting

$$v_0 := -\frac{1}{a^2 + 1}(e_1 + a^2f_1), \quad v_1 := \frac{1}{a^2 + 1}(a(e_1 - f_1) - a^2be_2 - bf_2), \\ v_2 := ae_2, \quad v_3 := f_2.$$

Thus, we have that

$$[v_0, v_1] = v_1, \quad [v_0, v_2] = -v_2, \quad [v_0, v_3] = -v_3,$$

$$Jv_0 = v_1, \quad Jv_2 = v_3,$$

and then  $\mathfrak{g} \cong \mathfrak{g}_1^h$ . The eigenspaces corresponding to  $E$ , in this new basis, are given by

$$\mathfrak{g}_+ = \text{span} \left\{ v_0 - av_1 - \frac{ab}{a^2+1}v_3, v_2 \right\}, \quad \mathfrak{g}_- = \text{span} \left\{ v_0 + dv_1 + \frac{b}{a^2+1}v_2, v_3 \right\}.$$

This complex product structure is equivalent to either  $\{J^{(1)}, E_{\theta,0}^{(1)}\}$  if  $b = 0$  or  $\{J^{(1)}, E_{\theta,1}^{(1)}\}$  if  $b \neq 0$ , in both cases with  $\cos \theta = \frac{1-a^2}{1+a^2}$ ,  $\sin \theta = -\frac{2a}{1+a^2}$ . Note that  $\theta \neq 0$  and  $\theta \neq \pi$  (since  $a \neq 0$ ).

$$(C2) \quad \nabla = \nabla^1 \text{ and } \nabla' = \nabla^2 \text{ or } \nabla = \nabla^2 \text{ and } \nabla' = \nabla^1.$$

In these cases there does not exist any  $\varphi : \mathfrak{u} \longrightarrow \mathfrak{v}$  compatible with  $\nabla$  and  $\nabla'$ .

$$(C3) \quad \nabla = \nabla^2 \text{ and } \nabla' = \nabla^2.$$

We are looking for a linear isomorphism  $\varphi : \mathfrak{u} \longrightarrow \mathfrak{v}$  compatible with  $\nabla$  and  $\nabla'$  and also with  $\omega$  and  $\omega'$ . After lengthy computations, we obtain that  $\varphi$  must be of the form  $\varphi = \begin{pmatrix} a & 0 \\ b & d \end{pmatrix}$ , with  $ad = 1$  (in the ordered bases  $\{e_1, e_2\}, \{f_1, f_2\}$ ). Hence, we have a hypersymplectic structure on the bicrossproduct Lie algebra  $\mathfrak{g} := \mathfrak{u} \bowtie \mathfrak{v} = \mathfrak{aff}(\mathbb{R}) \bowtie \mathfrak{aff}(\mathbb{R})$ . Let us describe this Lie algebra. Let us denote  $e_i := (e_i, 0)$ ,  $f_i := (0, f_i)$  for  $i = 1, 2$ ; the only non zero brackets are

$$\begin{aligned} [e_1, e_2] &= e_2, & [e_1, f_1] &= \frac{1}{2}e_1 - abe_2 - \frac{1}{2}f_1 - bdf_2, \\ [e_1, f_2] &= \frac{1}{2}(a^2e_2 + f_2), & [e_2, f_1] &= -\frac{1}{2}(e_2 + d^2f_2), & [f_1, f_2] &= f_2. \end{aligned}$$

The complex product structure on this Lie algebra is given by

$$Je_1 = af_1 + bf_2, \quad Je_2 = df_2, \quad Jf_1 = -de_1 + be_2, \quad Jf_2 = -ae_2$$

and  $Ee_i = e_i$ ,  $Ef_i = -f_i$  for  $i = 1, 2$ . We will make a change of basis, setting

$$\begin{aligned} v_0 &:= \frac{2}{a^2+1} \left( e_1 - \frac{2a^3b}{3(a^2+1)}e_2 + a^2f_1 + \frac{2ab}{3(a^2+1)}f_2 \right), \\ v_2 &:= \frac{2}{a^2+1} \left( a(e_1 - f_1) - \frac{a^2b(3a^2+1)}{3(a^2+1)}e_2 - \frac{b(a^2+3)}{3(a^2+1)}f_2 \right), \\ v_1 &:= e_2 + f_2, \quad v_3 := -ae_2 + df_2. \end{aligned}$$

Thus, we have that

$$[v_0, v_1] = 2v_1, \quad [v_0, v_2] = -v_2, \quad [v_0, v_3] = v_3, \quad [v_1, v_2] = v_3,$$

$$Jv_0 = -v_2, \quad Jv_1 = v_3,$$

and then  $\mathfrak{g} \cong \mathfrak{g}_2^h$ . The eigenspaces corresponding to  $E$ , in this new basis, are given by

$$\begin{aligned} \mathfrak{g}_+ &= \text{span} \left\{ v_0 + av_2 + \frac{2a^2b}{3(a^2+1)}v_3, v_1 - av_3 \right\}, \\ \mathfrak{g}_- &= \text{span} \left\{ v_0 - dv_2 - \frac{2ab}{3(a^2+1)}v_1, av_1 + v_3 \right\}. \end{aligned}$$

This complex product structure is equivalent to either  $\{J^{(2)}, E_{\theta,0}^{(2)}\}$  if  $b = 0$  or  $\{J^{(2)}, E_{\theta,1}^{(2)}\}$  if  $b \neq 0$ , in both cases with  $\cos \theta = \frac{1-a^2}{1+a^2}$ ,  $\sin \theta = -\frac{2a}{1+a^2}$ . Note that  $\theta \neq 0$  and  $\theta \neq \pi$  (since  $a \neq 0$ ).  $\square$

*Remark.* Complex structures on 4-dimensional solvable Lie algebras were classified in [13] and [15]. The Lie algebra  $\mathfrak{g}_0^h$  lies in the class  $S1$  of [15], the algebra  $\mathfrak{g}_1^h$  lies in the class  $A2$  ( $\lambda = -1$ ) of [13] and finally,  $\mathfrak{g}_2^h$  is in the class  $H5$  ( $\lambda_1 = 2$ ,  $\lambda_2 = -1$ ) of [13]. The first two Lie algebras carry only one complex structure, up to equivalence, and they coincide with the complex structures  $J^{(0)}$  and  $J^{(1)}$  constructed in Theorem 5.1, respectively. In contrast,  $\mathfrak{g}_2^h$  carries two non equivalent complex structures: one of them coincides with the complex structure  $J^{(2)}$  from Theorem 5.1, while the other one cannot be part of any hypersymplectic structure on  $\mathfrak{g}_2^h$ .

## 6. HYPERSYMPLECTIC METRICS ON THE ASSOCIATED LIE GROUPS

We will determine now the hypersymplectic metrics on the Lie algebras described in Theorem 5.1. We will show in the following lemma that given a complex product structure on a 4-dimensional Lie algebra, there is only one compatible metric, up to a non zero constant. A proof of this lemma can also be found in [4].

**Lemma 6.1.** *Let  $\{J, E\}$  be a complex product structure on a 4-dimensional Lie algebra. If  $g$  and  $h$  are two metrics on  $\mathfrak{g}$  compatible with  $\{J, E\}$ , then there exists  $\lambda \in \mathbb{R} \setminus \{0\}$  such that  $h = \lambda g$ .*

*Proof.* Since  $g$  and  $h$  are non degenerate, there exists a linear isomorphism  $T$  of  $\mathfrak{g}$  such that  $h(x, y) = g(Tx, y)$  for all  $x, y \in \mathfrak{g}$ . Let  $\lambda$  be an eigenvalue of  $T$ , which is a non zero real number, and let  $V_\lambda$  denote the corresponding eigenspace. Using equations (5), we obtain that  $T$  commutes with  $J$  and  $E$ ; therefore,  $V_\lambda$  is invariant by  $J$  and  $E$  and then the dimension of  $V_\lambda$  is even.

Let us suppose that  $\dim V_\lambda = 2$ . As  $V_\lambda$  is a proper subspace of  $\mathfrak{g}$ ,  $T$  has another eigenvalue  $\mu \neq \lambda$  ( $\mu \in \mathbb{R} \setminus \{0\}$ ) with  $V_\mu$ , the eigenspace corresponding to  $\mu$ , also invariant by  $J$  and  $E$ ; hence  $\mathfrak{g} = V_\lambda \oplus V_\mu$ . The metric  $g$  is degenerate on both  $V_\lambda$  and  $V_\mu$  since otherwise the dimension of each of these subspaces would be a multiple of 4; in fact  $g = 0$  on  $V_\lambda$  and on  $V_\mu$ .

There exist  $x \in V_\lambda$ ,  $y \in V_\mu$  such that  $g(x, y) \neq 0$ , since  $g$  is non degenerate. We compute

$$h(x, y) = g(Tx, y) = \lambda g(x, y)$$

and also

$$h(x, y) = h(y, x) = g(Ty, x) = \mu g(x, y).$$

Since  $g(x, y) \neq 0$ , we have that  $\lambda = \mu$ , a contradiction. Therefore  $\dim V_\lambda = 4$  and the lemma follows.  $\square$

*Remark.*



**6.1. The Lie algebra  $\mathbb{R}^4$ .** In this case, the corresponding simply connected abelian Lie group is the pseudo-Riemannian manifold  $\mathbb{R}^{2,2}$ , that is,  $\mathbb{R}^4$  with the neutral metric  $g = dt^2 + dx^2 - dy^2 - dz^2$ , where  $t, x, y, z$  are the global canonical coordinates on  $\mathbb{R}^4$ . This metric is complete and flat.

**6.2. The Lie algebra  $\mathfrak{g}_0^h$ .** Let  $G_0^h$  denote the simply connected Lie group corresponding to  $\mathfrak{g}_0^h$ . It is well known that  $G_0^h$  is diffeomorphic to  $\mathbb{R}^4$  and, by standard computations, we can find global coordinates  $t, x, y, z$  on  $G_0^h$  such that the left-invariant 1-forms  $\{v^0, v^1, v^2, v^3\}$  are given by

$$\begin{aligned} v^0 &= dt, \\ v^1 &= dx, \\ v^2 &= dy, \\ v^3 &= -x dy + dz. \end{aligned}$$

Let us consider the complex product structure  $\{J^{(0)}, E_\theta^{(0)}\}$  as given in Theorem 5.1. The subalgebras  $\mathfrak{g}_+$  and  $\mathfrak{g}_-$  corresponding to  $E_\theta^{(0)}$  are given by

$$\mathfrak{g}_+ = \text{span}\{\cos_{\theta/2} v_3 + \sin_{\theta/2} v_0, v_1\}, \quad \mathfrak{g}_- = \text{span}\{-\sin_{\theta/2} v_3 + \cos_{\theta/2} v_0, v_2\}.$$

Let us denote  $U_\theta = \cos_{\theta/2} v_3 + \sin_{\theta/2} v_0$  and  $V_\theta = -\sin_{\theta/2} v_3 + \cos_{\theta/2} v_0$ . Every hypersymplectic metric on  $\mathfrak{g}_1^h$  corresponding to this complex product structure is homothetic to

$$g_\theta = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

in the ordered basis  $\{U_\theta, v_1, V_\theta, v_2\}$ , due to Lemma 6.1. If  $\{v^0, v^1, v^2, v^3\}$  is the dual basis of  $\{v_0, v_1, v_2, v_3\}$ , then  $g_{\theta,0}$  can be written as

$$\begin{aligned} g_\theta &= -(\cos_{\theta/2} v^3 + \sin_{\theta/2} v^0) \otimes v^2 + v^1 \otimes (-\sin_{\theta/2} v^3 + \cos_{\theta/2} v^0) + \\ &\quad (-\sin_{\theta/2} v^3 + \cos_{\theta/2} v^0) \otimes v^1 - v^2 \otimes (\cos_{\theta/2} v^3 + \sin_{\theta/2} v^0), \end{aligned}$$

or, equivalently,

$$g_\theta = -(\cos_{\theta/2} v^3 + \sin_{\theta/2} v^0) \cdot v^2 + v^1 \cdot (-\sin_{\theta/2} v^3 + \cos_{\theta/2} v^0)$$

where  $\cdot$  denotes the symmetric product of 1-forms. Hence, the left-invariant metric  $g_\theta$  on  $G_0^h$  is given in terms of the global coordinates by

$$\begin{aligned} g_\theta &= -\cos_{\theta/2} x^2 dy^2 + \cos_{\theta/2} x dy dz - \cos_{\theta/2} dz^2 + \sin_{\theta/2} x dt dy - \\ &\quad \sin_{\theta/2} dt dz + \sin_{\theta/2} x dx dy - \sin_{\theta/2} dx dz + \cos_{\theta/2} dt dx. \end{aligned}$$

The torsion-free connection  $\nabla^{\text{CP}}$  on  $\mathfrak{g}_0^h$  associated to  $\{J^{(0)}, E_\theta^{(0)}\}$  (or the Levi-Civita connection of  $g_\theta$ ) is easy to compute, and we can readily verify that this connection is flat. Also, using equation (1), we obtain that this connection is complete. Hence, the metrics  $g_\theta$  on  $G_0^h$  are all flat and complete, and thus these metrics are all isometric to the canonical neutral metric on  $\mathbb{R}^4$ , even though the hypersymplectic structures are not equivalent.

**6.3. The Lie algebra  $\mathfrak{g}_1^h$ .** Let  $G_1^h$  denote the simply connected Lie group corresponding to  $\mathfrak{g}_1^h$ . It is well known that  $G_1^h$  is diffeomorphic to  $\mathbb{R}^4$  and, by standard computations, we can find global coordinates  $t, x, y, z$  on  $G_1^h$  such that the left-invariants 1-forms  $\{v^0, v^1, v^2, v^3\}$  are given by

$$\begin{aligned} v^0 &= dt, \\ v^1 &= e^{-t} dx, \\ v^2 &= e^t dy, \\ v^3 &= e^t dz. \end{aligned}$$

(i) Let us fix firstly the complex product structure  $\{J^{(1)}, E_{\theta,0}^{(1)}\}$  as given in Theorem 5.1. The subalgebras  $\mathfrak{g}_+$  and  $\mathfrak{g}_-$  corresponding to  $E_{\theta,0}^{(1)}$  are given by

$$\mathfrak{g}_+ = \text{span}\{\cos_{\theta/2} v_0 + \sin_{\theta/2} v_1, v_2\}, \quad \mathfrak{g}_- = \text{span}\{-\sin_{\theta/2} v_0 + \cos_{\theta/2} v_1, v_3\}.$$

Let us denote  $U_\theta = \cos_{\theta/2} v_0 + \sin_{\theta/2} v_1$  and  $V_\theta = -\sin_{\theta/2} v_0 + \cos_{\theta/2} v_1$ . Note that  $JU_\theta = V_\theta$  and  $Jv_2 = v_3$ . Every hypersymplectic metric on  $\mathfrak{g}_1^h$  corresponding to this complex product structure is homothetic to

$$g_{\theta,0} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

in the ordered basis  $\{U_\theta, v_2, V_\theta, v_3\}$ , due to Lemma 6.1. If  $\{v^0, v^1, v^2, v^3\}$  is the dual basis of  $\{v_0, v_1, v_2, v_3\}$ , then  $g_{\theta,0}$  can be written as

$$\begin{aligned} g_{\theta,0} &= -(\cos_{\theta/2} v^0 + \sin_{\theta/2} v^1) \otimes v^3 + v^2 \otimes (-\sin_{\theta/2} v^0 + \cos_{\theta/2} v^1) + \\ &\quad (-\sin_{\theta/2} v^0 + \cos_{\theta/2} v^1) \otimes v^2 - v^3 \otimes (\cos_{\theta/2} v^0 + \sin_{\theta/2} v^1) \end{aligned}$$

or, equivalently,

$$g_{\theta,0} = -(\cos_{\theta/2} v^0 + \sin_{\theta/2} v^1) \cdot v^3 + (-\sin_{\theta/2} v^0 + \cos_{\theta/2} v^1) \cdot v^2$$

where  $\cdot$  denotes the symmetric product of 1-forms. Hence, the left-invariant metric  $g_{\theta,0}$  on  $G_1^h$  is given in terms of the global coordinates by

$$g_{\theta,0} = -\cos_{\theta/2} e^t dt dz - \sin_{\theta/2} dx dz - \sin_{\theta/2} e^t dt dy + \cos_{\theta/2} dx dy.$$

The connection  $\nabla^{\text{CP}} = \nabla^{g_{\theta,0}}$  on  $\mathfrak{g}_1^h$  can be explicitly computed, and we can deduce from this computation that it is flat. However, this connection cannot be complete. Indeed, if it were complete, then its restrictions to the eigenspaces of  $E_{\theta,0}^{(1)}$  should be complete. But at least one of these eigenspaces is isomorphic to  $\mathfrak{aff}(\mathbb{R})$ , and we know from Proposition 4.4 that any flat torsion-free connection on  $\mathfrak{aff}(\mathbb{R})$  compatible with a symplectic form is not complete, a contradiction.

(ii) Let us fix now the complex product structure  $\{J^{(1)}, E_{\theta,1}^{(1)}\}$  as given in Theorem 5.1. The subalgebras  $\mathfrak{g}_+$  and  $\mathfrak{g}_-$  corresponding to  $E_{\theta,1}^{(1)}$  are given by

$$\begin{aligned} \mathfrak{g}_+ &= \text{span}\{\cos_{\theta/2} v_0 + \sin_{\theta/2} v_1 + \cos_{\theta/2} v_3, v_2\}, \\ \mathfrak{g}_- &= \text{span}\{-\sin_{\theta/2} v_0 + \cos_{\theta/2} v_1 - \cos_{\theta/2} v_2, v_3\}. \end{aligned}$$

Let us denote  $U_\theta = \cos_{\theta/2} v_0 + \sin_{\theta/2} v_1 + \cos_{\theta/2} v_3$  and  $V_\theta = -\sin_{\theta/2} v_0 + \cos_{\theta/2} v_1 - \cos_{\theta/2} v_2$ . Note that  $JU_\theta = V_\theta$  and  $Jv_2 = v_3$ . Every hypersymplectic metric on  $\mathfrak{g}_1^h$  corresponding to this complex product structure is homothetic to

$$g_{\theta,1} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

in the ordered basis  $\{U_\theta, v_2, V_\theta, v_3\}$ , because of Lemma 6.1. If  $\{v^0, v^1, v^2, v^3\}$  is the dual basis of  $\{v_0, v_1, v_2, v_3\}$ , then  $g_{\theta,1}$  can be written as

$$g_{\theta,1} = -(\cos_{\theta/2} v^0 + \sin_{\theta/2} v^1 + \cos_{\theta/2} v^3) \cdot v^3 + (-\sin_{\theta/2} v^0 + \cos_{\theta/2} v^1 - \cos_{\theta/2} v^2) \cdot v^2$$

where  $\cdot$  denotes the symmetric product of 1-forms. Hence, the left-invariant metric  $g_{\theta,1}$  on  $G_1^h$  is given in terms of the global coordinates by

$$g_{\theta,1} = \cos_{\theta/2} e^t dt dz + \sin_{\theta/2} dx dz + \cos_{\theta/2} e^{2t} dz^2 + \sin_{\theta/2} e^t dt dy - \cos_{\theta/2} dx dy + \cos_{\theta/2} e^{2t} dy^2.$$

The connection  $\nabla^{\text{CP}} = \nabla^{g_{\theta,1}}$  on  $\mathfrak{g}_1^h$  can be explicitly computed, and we can deduce from this computation that

$$R(U_\theta, V_\theta) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 6 \cos(\theta/2) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 6 \cos(\theta/2) & 0 \end{pmatrix}$$

in the ordered basis  $\{U_\theta, v_2, V_\theta, v_3\}$  and is zero for the other possibilities. Hence,  $g_{\theta,1}$  is flat if and only if  $\theta = \pi$ . As in the previous case, the metrics  $g_{\theta,1}$  are not complete.

(iii) Let us consider finally the complex product structure  $\{J^{(1)}, E_1^{(1)}\}$  as given in Theorem 5.1. The subalgebras  $\mathfrak{g}_+$  and  $\mathfrak{g}_-$  corresponding to  $E_1^{(1)}$  are

$$\mathfrak{g}_+ = \text{span}\{v_1 + v_3, v_2\}, \quad \mathfrak{g}_- = \text{span}\{-v_0 - v_2, v_3\}.$$

Every hypersymplectic metric on  $\mathfrak{g}_1^h$  corresponding to this complex product structure is homothetic to

$$g_1 = -(v^1 + v^3) \cdot v^3 - (v^0 + v^2) \cdot v^2,$$

which gives rise to a left-invariant metric on  $G_1^h$ , given by

$$g_1 = dx dz + e^{2t} dz^2 + e^t dt dy + e^{2t} dy^2.$$

It can be shown that the metric  $g_1$  is flat and not complete.

**6.4. The Lie algebra  $\mathfrak{g}_2^h$ .** Let  $G_2^h$  denote the simply connected Lie group corresponding to  $\mathfrak{g}_2^h$ . It is well known that  $G_2^h$  is diffeomorphic to  $\mathbb{R}^4$  and, by standard computations, we can find global coordinates  $t, x, y, z$  on  $G_2^h$  such that the left-invariants 1-forms  $\{v^0, v^1, v^2, v^3\}$

are given by

$$\begin{aligned} v^0 &= dt, \\ v^1 &= e^{-2t} dx, \\ v^2 &= e^t dy, \\ v^3 &= e^{-t}(dz - \tfrac{1}{2}x dy + \tfrac{1}{2}y dx). \end{aligned}$$

(i) Let us fix firstly the complex product structure  $\{J^{(2)}, E_{\theta,0}^{(2)}\}$  as given in Theorem 5.1. The subalgebras  $\mathfrak{g}_+$  and  $\mathfrak{g}_-$  corresponding to  $E_{\theta,0}^{(2)}$  are given by

$$\begin{aligned} \mathfrak{g}_+ &= \text{span}\{\cos_{\theta/2} v_0 + \sin_{\theta/2} v_2, \cos_{\theta/2} v_1 - \sin_{\theta/2} v_3\}, \\ \mathfrak{g}_- &= \text{span}\{\sin_{\theta/2} v_0 - \cos_{\theta/2} v_2, \sin_{\theta/2} v_1 + \cos_{\theta/2} v_3\}. \end{aligned}$$

Let us denote  $U_\theta = \cos_{\theta/2} v_0 + \sin_{\theta/2} v_1$ ,  $\tilde{U}_\theta = \cos_{\theta/2} v_1 - \sin_{\theta/2} v_3$  and  $V_\theta = -\sin_{\theta/2} v_0 + \cos_{\theta/2} v_1$ ,  $\tilde{V}_\theta = \sin_{\theta/2} v_1 + \cos_{\theta/2} v_3$ . Note that  $JU_\theta = V_\theta$  and  $J\tilde{U}_\theta = \tilde{V}_\theta$ . Every hypersymplectic metric on  $\mathfrak{g}_2^h$  corresponding to this complex product structure is homothetic to

$$g_{\theta,0} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

in the ordered basis  $\{U_\theta, \tilde{U}_\theta, V_\theta, \tilde{V}_\theta\}$ , due to Lemma 6.1. If  $\{v^0, v^1, v^2, v^3\}$  is the dual basis of  $\{v_0, v_1, v_2, v_3\}$ , then  $g_{\theta,0}$  can be written as

$$g_{\theta,0} = -(\cos_{\theta/2} v^0 + \sin_{\theta/2} v^2) \cdot (\sin_{\theta/2} v^1 + \cos_{\theta/2} v^3) + (\cos_{\theta/2} v^1 - \sin_{\theta/2} v^3) \cdot (\sin_{\theta/2} v^1 + \cos_{\theta/2} v^3)$$

where  $\cdot$  denotes the symmetric product of 1-forms. Hence, the left-invariant metric  $g_{\theta,0}$  on  $G_2^h$  is given in terms of the global coordinates by

$$g_{\theta,0} = e^{-t} dt(dz - \tfrac{1}{2}x dy + \tfrac{1}{2}y dx) + e^{-t} dx dy.$$

Note that  $g_{\theta,0}$  does not depend on  $\theta$ . It can be shown that this metric is flat and not complete.

(ii) Let us fix now the complex product structure  $\{J^{(2)}, E_{\theta,1}^{(2)}\}$  as given in Theorem 5.1. The subalgebras  $\mathfrak{g}_+$  and  $\mathfrak{g}_-$  corresponding to  $E_{\theta,1}^{(2)}$  are given by

$$\begin{aligned} \mathfrak{g}_+ &= \text{span}\{\cos_{\theta/2} v_0 + \sin_{\theta/2} v_2 + \cos_{\theta/2} v_3, \cos_{\theta/2} v_1 - \sin_{\theta/2} v_3\} \\ \mathfrak{g}_- &= \text{span}\{\sin_{\theta/2} v_0 - \cos_{\theta/2} v_2 - \cos_{\theta/2} v_1, \sin_{\theta/2} v_1 + \cos_{\theta/2} v_3\}. \end{aligned}$$

Let us denote  $U_\theta = \cos_{\theta/2} v_0 + \sin_{\theta/2} v_2 + \cos_{\theta/2} v_3$ ,  $\tilde{U}_\theta = \cos_{\theta/2} v_1 - \sin_{\theta/2} v_3$  and  $V_\theta = \sin_{\theta/2} v_0 - \cos_{\theta/2} v_2 - \cos_{\theta/2} v_1$ ,  $\tilde{V}_\theta = \sin_{\theta/2} v_1 + \cos_{\theta/2} v_3$ . Note that  $JU_\theta = V_\theta$  and  $J\tilde{U}_\theta = \tilde{V}_\theta$ . Every hypersymplectic metric on  $\mathfrak{g}_2^h$  corresponding to this complex product structure is homothetic to

$$g_{\theta,1} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

in the ordered basis  $\{U_\theta, \tilde{U}_\theta, V_\theta, \tilde{V}_\theta\}$ , because of Lemma 6.1. If  $\{v^0, v^1, v^2, v^3\}$  is the dual basis of  $\{v_0, v_1, v_2, v_3\}$ , then  $g_{\theta,1}$  can be written as

$$g_{\theta,1} = -(\cos_{\theta/2} v^0 + \sin_{\theta/2} v^2 + \cos_{\theta/2} v^3) \cdot (\sin_{\theta/2} v^1 + \cos_{\theta/2} v^3) + (\sin_{\theta/2} v^0 - \cos_{\theta/2} v^2 - \cos_{\theta/2} v^1) \cdot (\cos_{\theta/2} v^1 - \sin_{\theta/2} v^3)$$

where  $\cdot$  denotes the symmetric product of 1-forms. Hence, the left-invariant metric  $g_{\theta,1}$  on  $G_2^h$  is given in terms of the global coordinates by

$$g_{\theta,1} = e^{-t} dt(dz - \frac{1}{2}x dy + \frac{1}{2}y dx) + e^{-t} dx dy + \cos_{\theta/2}^2 e^{-4t} dx^2 + \cos_{\theta/2}^2 e^{-2t} (dz - \frac{1}{2}x dy + \frac{1}{2}y dx)^2.$$

It can be shown that the curvature of the connection  $\nabla^{\text{CP}} = \nabla^{g_{\theta,1}}$  is given by

$$R(U_\theta, V_\theta) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 6\cos^2(\theta/2) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 6\cos^2(\theta/2) & 0 \end{pmatrix}$$

in the ordered basis  $\{U_\theta, \tilde{U}_\theta, V_\theta, \tilde{V}_\theta\}$  and is zero for the other possibilities. Hence,  $g_{\theta,1}$  is flat if and only if  $\theta = \pi$ . As in previous cases, the metrics  $g_{\theta,1}$  are not complete.

(iii) Let us consider finally the complex product structure  $\{J^{(2)}, E_1^{(2)}\}$  as given in Theorem 5.1. The subalgebras  $\mathfrak{g}_+$  and  $\mathfrak{g}_-$  corresponding to  $E_1^{(2)}$  are

$$\mathfrak{g}_+ = \text{span}\{v_1 + v_2, v_3\}, \quad \mathfrak{g}_- = \text{span}\{v_0 + v_3, -v_1\}.$$

Every hypersymplectic metric on  $\mathfrak{g}_2^h$  corresponding to this complex product structure is homothetic to

$$g_1 = (v^1 + v^2) \cdot v^1 + (v^0 + v^3) \cdot v^3,$$

which gives rise to a left-invariant metric on  $G_2^h$ , given by

$$g_1 = e^{-t} dx dy + e^{-4t} dx^2 + e^{-t} dt(dz - \frac{1}{2}x dy + \frac{1}{2}y dx) + e^{-2t} (dz - \frac{1}{2}x dy + \frac{1}{2}y dx)^2.$$

This metric is flat and not complete.

## REFERENCES

- [1] A. Andrada, M. L. Barberis, I. Dotti and G. Ovando: Four-dimensional solvable Lie algebras. Preprint.
- [2] A. Andrada and S. Salamon: Complex product structures on Lie algebras. To appear in Forum Math.
- [3] J. Barret, G. W. Gibbons, M. J. Perry, C. N. Pope and P. Ruback: Kleinian geometry and the  $N = 2$  superstring. Int. J. Mod. Phys. A9 (1994), 1457–1494
- [4] N. Blazić and S. Vukmirović: Four-dimensional Lie algebras with a para-hypercomplex structure. Available at arXiv: math.DG/0310180.
- [5] A. Fino, H. Pedersen, Y. S. Poon and M. Sørensen: Neutral Calabi-Yau structures on Kodaira manifolds. Preprint, 2002: <http://www.imada.sdu.dk>
- [6] M. Guediri: Sur la complétude des pseudo-métriques invariantes à gauche sur les groupes de Lie nilpotentes. Rend. Sem. Mat. Univ. Pol. Torino **52** (1994), 371–376
- [7] N. Hitchin: Hypersymplectic quotients. Acta Academiae Scientiarum Tauriensis, Supplemento al numero **124** (1990), 169–180
- [8] H. Kamada: Neutral hyperkähler structures on primary Kodaira surfaces. Tsukuba J. Math. **23** No. 2 (1999), 321–332
- [9] S. Kaneyuki and M. Kozai: Paracomplex structures and affine symmetric spaces. Tokyo J. Math. **8** (1985), 81–98
- [10] P. Libermann: Sur les structures presque paracomplexes. C. R. Acad. Sci. Paris **234** (1952), 2517–2519

- [11] S. Majid: Matched pairs of Lie groups associated to solutions of the Yang-Baxter equations. *Pacific J. Math.* **141** (1990), 311–332
- [12] A. Masuoka: Extensions of Hopf algebras and Lie bialgebras. *Trans. Amer. Math. Soc.* **352** (2000), 3837–3879
- [13] G. Ovando: Invariant complex structures on solvable real Lie groups. *Manuscripta Math.* **103** (2000), 19–30
- [14] E. Remm and M. Goze: Affine structures on abelian Lie groups. *Linear Algebra Appl.* **360** (2003), 215–230
- [15] J. E. Snow: Invariant complex structures on four-dimensional solvable real Lie groups. *Manuscripta Math.* **66** (1990), 397–412
- [16] S. Vukmirović: Paraquaternionic reduction. Available at arXiv: math.DG/0304424

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